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Propagation of electromagnetic waves from an arbitrary source.

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PROPAGATION OF ELECTROMAGNETIC WAVES FROM AN ARBITRARY SOURCE THROUGH INHOMOGENEOUS STRATIFIED ATMOSPHERES

by

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1. Introduction.

This paper offers a new approach to the problem of determining the field propagated in a non-homogeneous atmosphere by an arbitrary source of electromagnetic radiation. The principal problem in connection with propagation of ultra-high frequency radiation is to determine the refractive effect arising from inhomogeneities in the atmosphere. Refraction not only distorts the radiation pattern of the source but produces singular or anomalous effects such as propagation beyond the optical horizon and trapping of radiation at definite levels so that the intensity at a distant point within such a level is far greater than would exist from the normal spreading of radiation in a homogeneous atmosphere. Even the normal atmosphere produces the effect of "bending" radiation down beyond the optical horizon. However, the effect of a normal atmosphere, in which the index of refraction decreases linearly with height above the earth's surface, can be taken into account by increasing the earth's radius by a factor of 4/3 and then treating the atmosphere as homogeneous[1]. On the other hand the departures from a normal atmosphere produced by changes in temperature, pressure, and water-vapor content, which seriously affect ultra-high frequency radio wave propagation, are not readily evaluated. The existence and magnitudo of these abnormal effects have now been noted so often experimentally[2] that the problem of determining theoretically the field produced by the inhomogeneities of the atmosphere has become an important one.

This paper approaches the theoretical problem in the following manner. It assumes, as do other investigations, that the electromagnetic parameters, ε, μ, and σ vary only with height above the earth but not in any direction parallel to the surface of the earth. It assumes further that the earth is flat. (Correction for this assumption and the accuracy of this correction factor are discussed in article 2.) Under these assumptions the paper shows, first that solutions of Maxwell's equations can be obtained through two scalar functions V and W which satisfy certain partial differential equations of the second order, [Equations (5.2)]. The functions V and W are essentially the analogues in a stratified medium of the Hertz vector potential which serves so effectively in media with constant ε, μ, and σ.

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1. Schelling, J.C., Burrows, C.R., and Ferrel, E.B.: Ultra-short Wave Propagation, Proc. I.R.E., V. 21, 1933, pp 427-463. Also Eckersley, T.L.: Ultra-short Wave Refraction and Diffraction, Jour. Inst. Elec. Eng., V. 80, 1937, pp. 286-304.

2. Many of the references cited later in other connections give experimental data. Systematic experimentation is being done also by the Georgia School of Technology.

When the source is a plane wave, expressions for V and W which satisfy the partial differential equations are, at least theoretically, readily obtained. To treat next any arbitrary source of electromagnetic radiation it is supposed that the source can be expressed in a homogeneous medium as a superposition of ordinary plane waves, [Expression (5.7)] . This supposition is certainly correct for the most important single type of source, namely the electric dipole, and for many others. By a mathematical argument which is fundamentally simple it is then shown that the radiation field produced in the non-homogeneous medium by an arbitrary source can be obtained by a superposition of the radiation fields produced by the plane waves which comprise the source. The amplitudes with which the individual plane waves enter into the mathematical representation of the field of the arbitrary source when in a homogeneous medium, of course enter into the determination of the field in the non-homogeneous medium. That is to say, the property which distinguishes one source from another resides in an amplitude function which depends upon the directions of the constituent plane waves.

This argument is applied first to the case where a source is located below a semi-infinite non-homogeneous medium and the field in the medium is then determined. The more general case of an arbitrary source located between two semi-infinite media is treated next. This section can be specialized to treat the case of a source of radiation located above the earth. The main result of the paper is embodied in formulas (7.8) and (7.9), which give the expressions for V and W. These expressions, taken in conjunction with formulas (5.3), permit determination of the six components of the electromagnetic field from V and W. A number of special cases are worked out to illustrate the theory, although they do not necessarily represent any actual atmospheric conditions.

It will be seen that the main result, [Equations (7.8) and (7.9)], is in the form of a double integral of a complex function. The problem of evaluating this integral for any particular atmospheric law of variation of index of refraction with height can be a difficult one. Some of the examples of the paper illustrate this point. However, difficulties in the evaluation of the integral are to be expected. As our discussion below will indicate, the one satisfactory approach which has thus far been made to the problem of anomalous propagation has produced tremendous calculation problems. The problem of evaluation of the integral may be considerably simplified by the use of reasonable assumptions. For example, it is practical to use the fact that the variation of the index of refraction with wave length is very small. As a matter of fact this assumption is made by other investigators at the very outset of their work.

The present approach to the problem of anomalous propagation is an alternative to the approach initiated and developed by several workers in the field, notably Furry, Hartree, Pearcey, Pryce and Pekeris[3]. The latter approach may be described briefly as follows. Under the assumption that the variation in the index of refraction, n, is small compared to the wave length, λ, the steady state electromagnetic field can be obtained from the function $\psi(x,y,z)$ which satisfies the wave equation

$$\nabla^2 \psi + k^2 n^2 \psi = 0 \tag{1.1}$$

where $k = \frac{2\pi}{\lambda}$. For a vertical dipole ψ represents the vertical component of the electrical Hertzian potential function of the dipole, and for a horizontal dipole ψ represents the vertical component of the magnetic Hertzian potential function of the dipole. (It can be shown that the field intensity $|E|$ is proportional to $|\psi|$ in each case.)

This equation must be solved for ψ subject to several boundary conditions which arise from the fact that the source is a dipole, that the ground is a perfect conductor, and that the wave must be outgoing at infinity. It is then assumed that the earth is flat, correction being made for this assumption by the use of a modified index of refraction (see Art. 2).

The solution of equation (1.1) is obtained in the form of an infinite series

$$\psi = -i\pi \sum_{1}^{\infty} H_0^2 (k_m d)\, U_m(h_1)\, U_m(h_2) \tag{1.2}$$

where d is horizontal distance from the source and h_1 and h_2 are the heights of the transmitter and receiver above ground. The functions U_m and the quantities k_m must satisfy the differential equation

$$\frac{d^2 U_m}{dh^2} + (k^2 M^2 - k_m^2)\, U_m = 0, \tag{1.3}$$

where M is now the modified index of refraction as a function of height h above a flat earth and U_m must satisfy some boundary conditions arising out of the conditions on ψ. Because of these boundary conditions the values of k_m are limited to certain characteristic values. Hence only certain characteristic functions U_m, each with its corresponding k_m, may be used.

[3] The essential theory of this approach is given by Furry W.H.: Theory of Characteristic Functions in Problems of Anomalous Propagation, Radiation Laboratory Report #680, Feb. 1945.

This alternative method calls for finding the characteristic functions and characteristic values of the second order, ordinary differential equation (1.3) which cannot be solved exactly, except in very special cases which do not happen to represent real atmospheric conditions. The determination of the characteristic values and characteristic functions is a major problem of calculation and calls for the use of advanced mathematical methods. Because of the complexity of the problem many separate methods have been investigated and compared. Among the methods tried there are the phase integral, or B.W.K. method; the perturbation method which has been found to be applicable to complex eigenvalues in which the imaginary part is large (this corresponds physically to the case where the energy represented by the terms of the above series leaks copiously from the ducts formed in the atmosphere); Rayleigh's method, which appears to be applicable best to real eigenvalues (physically this corresponds to trapped modes); the variational method which seems to work well for real and complex eigenvalues which correspond to what may be physically described as somewhat leaky modes; and finally, the differential analyzer, which merely resorts to a machine to handle the labor and reduce the time required. Some discussion of these methods will be found in papers by G.G. Macfarlane[4] and C. L. Pekeris[5].

It is fair to say that all these methods have been only partially successful in that any one method works for only very special laws of variation of N with height and then only at the cost of tremendous calculations. For some laws of variation no methods have been successful in getting good approximations to the characteristic values. Indeed, to secure approximations to which one can attach some degree of assurance, it appears to be necessary, since the correct values are not known, to use several methods of calculation simultaneously and to rely upon agreement of the results, if such agreement is obtained.

When the characteristic values and characteristic functions are obtained to some degree of approximation they must be substituted in equation (1.2) and the series there summed. This summation is often feasible to a useful degree of approximation only in some ranges of d values.

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4. Macfarlane, G.G.: A Variational Method for Determining Eigenvalues of the Wave Equation Applied to Refraction. Cambridge Philosophical Society Proc., V. 43 Part II, April, 1947, pp. 213-219.

5. Pekeris, C.L.: Wave Theoretical Interpretation of Propagation of 10 Cm. and 3 Cm. Waves in Low-Level Ocean Ducts, Proc. I.R.E., V. 35, May 1947, pp. 453-462.

The theory of this paper will, as remarked above, likewise entail calculation problems. At the present time, all that can be said is that it has been possible in other problems of applied mathematics to obtain good approximations to the values of integrals by familiar methods, such as the method of steepest descent[6], where series solutions have led to tremendously long calculations and to very slow convergence.

The theory of this paper is considerably more general than other approaches in that the radiation source is arbitrary and the medium below the source is arbitrary. Other investigations have been limited to the dipole source for the reason that the propagated field is generally desired at points near the ground and at large distances from the transmitter. It is known that radiation in non-homogeneous atmospheres reaching distant points near the ground emanates from the source in directions deviating less than a degree from the horizontal. Hence it is argued that by computing for the dipole the ratio of energy per unit area received to energy per unit area passing horizontally at some fixed distance from the transmitter, one could compute the field received from any transmitter by measuring or calculating the energy produced at the fixed distance. However, there are many applications of a theory of propagation to which this argument does not apply. For example, to obtain the coverage pattern of a radar station for purposes of airplane detection, communications, and landing systems, requires more than a knowledge of the field propagated along the surface of the earth. That the coverage pattern of a radar set can be considerably modified by anomalous atmospheric conditions, has been stressed by Freehafer.[7]

The remarks of the last few paragraphs indicate reasons for undertaking a new approach rather than for pursuing further an approach which has been somewhat explored already.

It should be noted that a theory of propagation through stratified media has application to the design of interference films (as on glass) so as to eliminate reflection or to eliminate elliptical polarization from totally reflecting surfaces.

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6. See, for example, Stratton, J. A.: Electromagnetic Theory, McGraw Hill Book Co., Inc., N. Y., 1941, p. 368.

7. Freehafer, John E.: The Effect of Atmospheric Refraction on Short Radio Waves, Radiation Laboratory Report #447.

2. The Use of a Flat Earth

In this paper, as in other investigations of the problem of propagation of high frequency waves in non-homogeneous atmospheres, it is supposed that the surface of the earth is flat. The errors introduced thereby are corrected somewhat by a device now standard known as the modified index of refraction. As remarked in article 1, it is assumed commonly that the index of refraction varies only with height above the surface of the earth. One then replaces the variation of the index of refraction with height above the surface of the spherical earth by a modified index of refraction which is given by the formula

$$M(r) = \frac{rn(r)}{bn(b)} \qquad (2.1)$$

where $n(r)$ is the actual variation of the index n with r, the distance from the earth's center, and b is any fixed value of r usually taken to be a, the earth's radius. If we replace r by $a + z$ in formula (2.1), since a is constant, we have a variation of index of refraction with z alone, z now being the height above the earth's surface in a rectangular coordinate system. In this coordinate system x and y represent distances along the flat surface. Any point on this surface corresponds to the point on earth having the same direction and the same distance (along the great circle route on the earth's surface) from the origin.

Physically expressed, the justification for the modified index of refraction is roughly that, instead of having the earth curve away from the radiation, the index is changed in such a way as to make the rays curve away from the earth. The mathematical justification for the use of this modified index of refraction, which corrects only approximately for the "flattening" of the earth, is given most clearly in a paper by Pekeris[8], wherein estimates of the validity of the approximation are also given. Essentially the approximation is valid out to ranges of the order of the radius of the earth, for wave lengths larger than a centimeter, and to heights of the order of 1,000 feet. If the range or height is exceeded or the wave length is decreased the approximation becomes poorer. Actual data are given by Pekeris in his report. The earth-flattening approximation is considered for purposes of ray-tracing by Freehafer.[9]

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8. Pekeris, C. L.: Accuracy of the Earth-Flattening Approximation, Col. U. Math. Phys. Group Report #3, Apr. 15, 1946. Also in Phys. Rev. V. 70, 1946, pp. 518-522,

9. Freehafer, John E.: loc.cit., see page 5 in particular.

It is also necessary in the case where the earth is treated as an imperfect conductor to correct for the index of refraction in the earth. For a conductor the index of refraction n is given by $\sqrt{\varepsilon_c \mu}$ where $\varepsilon_c = \quad - \frac{i\sigma}{\omega}$, and this n must be modified by altering ε and σ to produce relation (2.1). No correction is made for μ, for it is practically one in all atmospheres near the surface of the earth and for short distances in the earth.

The theory of this paper and the illustrative examples deal directly with ε, μ, and σ rather than with the index of refraction. No mention is made as to whether these values are modified or not. In the application of the theory to anomalous propagation over a flat earth, ε and σ would be modified, whereas in the application to a problem of interference layers they would not be. In the former case some of the illustrations of the theory given in this paper must be properly interpreted. For example if the medium used for purposes of illustration is supposed to consist of plane parallel layers within which ε, μ, and σ are constant, this situation over a flat earth corresponds to one in which the actual variation of the index of refraction with height above a spherical earth is of the form $\frac{k}{r}$ with k different from layer to layer. Just which cases of actual variation of n with height above a spherical earth may be usefully approximated by a series of functions of the form $\frac{k}{r}$ is, for the moment, irrelevant. In fact, the theory does not require that an actual atmosphere be treated by approximating it with a finite number of layers of definite thickness.

Because the approximation introduced by the modified index of refraction is not good for wave lengths less than 1 cm or for heights above about 1,000 feet, it seems advisable for these ranges to develop a theory for a spherical earth directly, and avoid this earth flattening approximation entirely. Some throught has already been given to a theory of propagation for a stratified medium above a spherical earth. Stratification in this case means that ε, μ, and σ are functions of r only where $r = \sqrt{x^2 + y^2 + z^2}$.

3. Some Theory Concerning Maxwell's Equations.

The electromagnetic vectors

$$E = (E_1, E_2, E_3)$$
$$H = (H_1, H_2, H_3)$$

satisfy Maxwell's equations

$$\frac{\varepsilon}{c} E_t - \text{curl } H = \frac{4\pi\sigma}{c} E \qquad (3.1)$$

$$\frac{\mu}{c} H_t + \text{curl } E = 0$$

Under the influence of a periodic source a field E, H is established which is periodic in time and can be represented in the form

$$E = \text{Real Part of } u\, e^{-i\omega t}$$
$$H = \text{Real Part of } v\, e^{-i\omega t} \qquad (3.2)$$

where the complex vectors u and v satisfy the equations

$$\text{curl } v + i\,k\,\varepsilon^* u = 0$$
$$\text{curl } u - i\,k\,\mu\, v = 0 \qquad (3.3)$$

where $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$ and $\varepsilon^* = \varepsilon - \frac{4\pi\sigma}{\omega} i$.

We shall assume in the following that $\sigma = 0$ and thus discuss only the equations

$$\frac{\varepsilon}{c} E_t - \text{curl } H = 0$$
$$\frac{\mu}{c} H_t + \text{curl } E = 0 \qquad (3.4)$$

We note, however, that from a time-periodic solution E, H of these equations a correct periodic solution of equation (3.1) with $\sigma \neq 0$ can be found simply by replacing the real quantity ε by the complex quantity $\varepsilon^* = \varepsilon - \frac{4\pi\sigma}{\omega} i$.

At any point x,y,z where there is no source the electromagnetic vectors E and H satisfy the additional equations, div ε H = div μH = 0, so that the complete system of equations is

$$\text{a) } \frac{\varepsilon}{c} E_t - \text{curl } H = 0, \qquad \text{c) div } \varepsilon E = 0$$
$$\text{b) } \frac{\mu}{c} H_t + \text{curl } E = 0, \qquad \text{d) div } \mu H = 0 \qquad (3.5)$$

By using the fact that the divergence of the curl of any vector function is 0, it is seen that the second set of equations is a consequence of the first if div εE = div $\mu H = 0$ at any particular time t. In case of time-periodic solutions, the latter two equations are a direct consequence of (3.3).

We can replace the equations (3.5) by second-order equations for the components of the vector E or of the vector H. By taking the curl of equation b) and substituting it in the time derivative of equation a) we obtain

$$\frac{\varepsilon\mu}{c^2} E_{tt} + \mu \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E = 0$$

Similarly, we obtain (3.6)

$$\frac{\varepsilon\mu}{c^2} H_{tt} + \varepsilon \operatorname{curl} \frac{1}{\varepsilon} \operatorname{curl} H = 0$$

If a solution E of the first equation is known for which div $\varepsilon E = 0$, then the associated magnetic vector H can be found by $\mu H_t = -c \operatorname{curl} E$. Similarly, if a solution H of the second equation is known for which div $\mu H = 0$, then E follows from

$$\varepsilon E_t = c \operatorname{curl} H.$$

With the aid of the vector identities[10]

$$\mu \operatorname{curl} \frac{1}{\mu} \operatorname{curl} E = - \Delta E + \operatorname{grad} \operatorname{div} E + \mu (\operatorname{grad} \frac{1}{\mu}) \times \operatorname{curl} E$$

and
$$\frac{1}{\varepsilon} \operatorname{div} \varepsilon E = 0 = \operatorname{div} E + \frac{1}{\varepsilon} (\operatorname{grad} \varepsilon) \cdot E$$

we can write equations (3.6) in the form

$$\frac{\varepsilon\mu}{c^2} E_{tt} - \Delta E = \operatorname{grad} (p \cdot E) + q \times \operatorname{curl} E \tag{3.7}$$

$$\frac{\varepsilon\mu}{c^2} H_{tt} - \Delta H = \operatorname{grad} (q \cdot H) + p \times \operatorname{curl} H$$

where the vectors p and q are defined by

$$p = \operatorname{grad} (\log \varepsilon)$$

$$q = \operatorname{grad} (\log \mu) \tag{3.8}$$

Note that the right sides of equation (3.7) contain only derivatives of first order.

In general, the six equations (3.7) involve all components in each equation. The simplication in the case of a stratified medium is that two equations can be obtained, one involving only one component of E, and the other only one component

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10. See Stratton, loc. cit., p. 604, relations (9) and (10).

of H. These components are the z-components E_3 and H_3 in the case of a stratified medium. This is shown in the next section.

4. Representation of E and H in a Stratified Medium.

We now specialize the results of article 3 to the case of a stratified medium. By such a medium we mean one in which ε, μ, and σ, vary in one dimension only. Since this paper replaces the spherical earth by a flat one by making the customary change in index of refraction, the direction in which ε, μ, and σ vary will be taken to be perpendicular to the flat earth and this direction indicated by the z-coordinate of our rectangular coordinate system. It must now be understood, however, that ε, μ, and σ are the modified characteristics of the atmosphere in accordance with the discussion of article 2.

We have in this case, from equation (3.8),

$$p = \frac{\varepsilon'}{\varepsilon} k \; ; \quad q = \frac{\mu'}{\mu} k$$

where primes denote differentiation with respect to z, and k is the unit vector in the z-direction.

Hence, from equations (3.7),

$$\frac{\varepsilon\mu}{c^2} E_{tt} - \Delta E = \text{grad}\left(\frac{\varepsilon'}{\varepsilon} E_3\right) + \frac{\mu'}{\mu}(k \times \text{curl } E)$$

$$\frac{\varepsilon\mu}{c^2} H_{tt} - \Delta H = \text{grad}\left(\frac{\mu'}{\mu} H_3\right) + \frac{\varepsilon'}{\varepsilon}(k \times \text{curl } H) \qquad (4.1)$$

In particular, by equating the third components of these two vector equations, we find that E_3 and H_3 satisfy, respectively,

$$\frac{\varepsilon\mu}{c^2} E_{3tt} - \Delta E_3 = \frac{\partial}{\partial z}\left(\frac{\varepsilon'}{\varepsilon} E_3\right)$$

$$\frac{\varepsilon\mu}{c^2} H_{3tt} - \Delta H_3 = \frac{\partial}{\partial z}\left(\frac{\mu'}{\mu} H_3\right) \qquad (4.2)$$

Our next problem is to determine the remaining components E_1, E_2, H_1 and H_2. For this purpose we use the equations div $\varepsilon E = 0$, div $\mu H = 0$ and the relations obtained for the components E_3 and H_3 from the first two of equations (3.5),

These four relations are:

$$\frac{\partial E_1}{\partial x} + \frac{\partial E_2}{\partial y} = -\frac{1}{\varepsilon}\frac{\partial(\varepsilon E_3)}{\partial z} \; ; \quad \frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = -\frac{1}{\mu}\frac{\partial(\mu H_3)}{\partial z}$$

$$\frac{\partial E_1}{\partial y} - \frac{\partial E_2}{\partial x} = \frac{\mu}{c}\frac{\partial H_3}{\partial t} \; ; \quad \frac{\partial H_1}{\partial y} - \frac{\partial H_2}{\partial x} = -\frac{\varepsilon}{c}\frac{\partial E_3}{\partial t} \qquad (4.3)$$

Solutions of these equations can be found as follows. We assume that E_3 and H_3 can be represented by two functions $V(x,y,z,t)$ and $W(x,y,z,t)$, as follows:

$$\varepsilon E_3 = -\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$
$$\mu H_3 = -\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) \tag{4.4}$$

This assumption does not introduce any loss of generality, since to any given pair E_3, H_3 many functions V, W can be constructed so that equation (4.4) is satisfied.

By introducing equations (4.4) in equations (4.2) we find that if V and W satisfy the equations

$$\frac{\varepsilon\mu}{c^2} V_{tt} = V_{xx} + V_{yy} + \varepsilon\left(\frac{1}{\varepsilon} V_z\right)_z$$
$$\frac{\varepsilon\mu}{c^2} W_{tt} = W_{xx} + W_{yy} + \mu\left(\frac{1}{\mu} W_z\right)_z \tag{4.5}$$

then the values of E_3 and H_3 given by relations (4.4) will satisfy equations (4.2).

With the representation (4.4) of E_3 and H_3 we now get solutions of equations (4.3) by

$$\varepsilon E_1 = V_{xz} - \frac{\varepsilon}{c} W_{yt} \qquad \mu H_1 = W_{xz} + \frac{\mu}{c} V_{yt}$$
$$\varepsilon E_2 = V_{yz} + \frac{\varepsilon}{c} W_{xt} \qquad \mu H_2 = W_{yz} - \frac{\mu}{c} V_{xt} \tag{4.6}$$

as one easily verifies by direct substitution.

On the assumption that Maxwell's equations hold we were led to equations (4.4), (4.5), and (4.6). For the purpose of this paper we must show next that the two vectors E and H defined by equations (4.4) and (4.6) satisfy Maxwell's equations; that is, we must show that we can reverse our steps.

Theorem. _If V and W are solutions of the following partial differential equations, wherein ε and μ are functions of z only,_

$$\frac{\varepsilon\mu}{c^2} V_{tt} = V_{xx} + V_{yy} + \varepsilon\left(\frac{1}{\varepsilon} V_z\right)_z$$
$$\frac{\varepsilon\mu}{c^2} W_{tt} = W_{xx} + W_{yy} + \mu\left(\frac{1}{\mu} W_z\right)_z \tag{4.7}$$

then a solution E, H of Maxwell's equations is obtained by the formulae:

$$\begin{aligned}
\varepsilon E_1 &= V_{xz} - \frac{\varepsilon}{c} W_{yt}\ , & \mu H_1 &= W_{xz} + \frac{\mu}{c} V_{yt} \\
\varepsilon E_2 &= V_{yz} + \frac{\varepsilon}{c} W_{xt}\ , & \mu H_2 &= W_{yz} - \frac{\mu}{c} V_{xt} \qquad (4.8)\\
\varepsilon E_3 &= - V_{xx} - V_{yy}\ , & \mu H_3 &= - W_{xx} - W_{yy}
\end{aligned}$$

Proof. We note that (4.8) may be written in vector form

$$\begin{aligned}
\varepsilon E &= \text{curl } A \\
\mu H &= \text{curl } B
\end{aligned} \qquad (4.9)$$

where A and B are the vectors

$$\begin{aligned}
A &= (V_y, - V_x, - \frac{\varepsilon}{c} W_t) \\
B &= (W_y, - W_x, \frac{\mu}{c} V_t)
\end{aligned} \qquad (4.10)$$

Since the divergence of the curl of any vector function is zero, it is evident that div εE = div $\mu H = 0$.

Furthermore, from equations (4.9) and (4.10)

$$\begin{aligned}
\frac{\varepsilon}{c} E_t - \text{curl } H &= \text{curl } (\frac{1}{c} A_t - H) \\
\frac{\mu}{c} H_t + \text{curl } E &= \text{curl } (\frac{1}{c} B_t + E)
\end{aligned} \qquad (4.11)$$

and by equations (4.8) and (4.10)

$$\frac{1}{c} A_t - H = (- \frac{1}{\mu} W_{xz}, - \frac{1}{\mu} W_{yz}, - \frac{\varepsilon}{c^2} W_{tt} + \frac{1}{\mu} W_{xx} + \frac{1}{\mu} W_{yy})$$

$$= \left(- \frac{1}{\mu} W_{xz}, - \frac{1}{\mu} W_{yz}, - (\frac{W_z}{\mu})_z\right)$$

by the second of relations (4.7);

$$= \text{grad } (- \frac{W_z}{\mu}) \ .$$

Again by relations (4.8) and (4.10)

$$\frac{1}{c} B_t + E = (\frac{V_{xz}}{\varepsilon}, \frac{V_{yz}}{\varepsilon}, \frac{\mu}{c^2} V_{tt} - \frac{1}{\varepsilon} V_{xx} - \frac{1}{\varepsilon} V_{yy}) \ .$$

By the first of relations (4.7) this

$$= \text{grad}\,(\frac{V_z}{\varepsilon}) \quad .$$

Since the curl of the gradient of any vector function is zero, we have by equation (4.11)

$$\frac{\varepsilon E_t}{c} - \text{curl } H = 0$$

$$\frac{\mu}{c} H_t + \text{curl } E = 0$$

which proves our theorem.

Our above consideration does not show, of course, that every solution, E, H, of Maxwell's equations, for which div εE = div $\mu H = 0$ and wherein ε and μ are functions of z only, can be represented in the form of equation (4.8). However, it is not difficult to see that this is the case. It is made plausible by the following reasoning.

The solution of Maxwell's equations is uniquely determined by assigning initial values to the six components of E and H at t = 0. However, these initial values must satisfy the condition div εE = div $\mu H = 0$, which reduces the "degree of freedom" in essence to four arbitrary functions. Exactly four functions, however, determine the solutions V and W of equation (4.7) namely the values of V, W, V_t, and W_t at t = 0.[11]

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11. Our formulas contain as a special case for $\varepsilon = \mu = 1$ and $W \equiv 0$ the Hertzian radiation from a dipole.

Letting $V = \frac{e^{i(r - \omega t)}}{r}$ we find from equations (4.10), (4.9) ~~(4.11)~~ and (4.7) ~~(4.12)~~ that

$$E = \text{curl }(\underline{i}V_y - \underline{j}V_x)$$

$$H = \frac{1}{c}\text{ curl } V_t\underline{k} = \text{curl } \pi_t \;,\; \pi = \frac{1}{c} V\underline{k}.$$

π is the Hertz vector of a dipole.

That the field of a more general Hertzian vector π can be represented by equation (4.9) ~~(4.12)~~ follows from the fact that curl grad$\varphi = 0$ and that to a given vector π a vector $\pi' = \pi + \text{grad}\varphi$ can be constructed such that π' has the special form $(W_y \,,\, -W_x \,,\, \frac{\mu}{c} V_t)$.

The theorem of this article deserves special attention. It says that for stratified media, i.e., media in which ε, μ, and σ functions of z only, the solution of Maxwell's equations, which involve 6 functions with at least two functions in each equation, can be reduced to the problem of finding two functions, namely E_3 and H_3, with a separate equation for each. This result is reached in equations (4.2). Additional theory then reduces the problem of finding E_3 and H_3 to that of solving equations (4.7), one for V and one for W. Formulas (4.8) then give the remaining four functions.

Standard electromagnetic theory permits the reduction of the problem of solving Maxwell's equations to the solution of a system of differential equations each involving only one function only in the case that ε, μ, and σ are constant. Hence the result of this paper extends standard theory at least to stratified media.

5. Solutions for V and W in the case of a stratified medium.

We have seen in the preceding sections that the electromagnetic field in a stratified medium can be represented with the aid of two scalar functions V and W. These functions satisfy the partial differential equations

$$\frac{\varepsilon\mu}{c^2} V_{tt} = V_{xx} + V_{yy} + \varepsilon\left(\frac{1}{\varepsilon} V_z\right)_z \tag{4.7}$$

$$\frac{\varepsilon\mu}{c^2} W_{tt} = W_{xx} + W_{yy} + \mu\left(\frac{1}{\mu} W_z\right)_z$$

The electromagnetic vectors E and H then are obtained by the formulae (4.8). We proceed to obtain functions V and W which satisfy these equations.

We do not assume in the following that $\varepsilon(z)$ and $\mu(z)$ are continuous functions, though we shall require that they are sectionally smooth. At any surface of discontinuity the functions E_1, E_2, εE_3 and H_1, H_2, μH_3 must be continuous. We recognize by relations (4.8) that this is the case if

<u>the functions V and W and the functions $\frac{1}{\varepsilon} V_z$ and $\frac{1}{\mu} W_z$ are continuous</u> (5.1)

We shall therefore seek a V and W which satisfy these conditions.

We shall confine ourselves in this section to solutions E and H which are periodic in time and thus have the form

$$R\left\{E\, e^{-i\omega t}\right\} \quad \text{and} \quad R\left\{H\, e^{-i\omega t}\right\}$$

where $E(x,y,z)$ and $H(x,y,z)$ are complex vectors independent of t. We obtain such solutions with the aid of functions V, W which have the form $V(x,y,z)e^{-i\omega t}$, $W(x,y,z)e^{-i\omega t}$, where V and W now are independent of t and satisfy the equations[12] obtained from equation (4.7) by substituting these new forms for the V and W there.

$$\begin{aligned} V_{xx} + V_{yy} + \varepsilon\left(\frac{1}{\varepsilon} V_z\right)_z + k^2 n^2 V &= 0 \\ W_{xx} + W_{yy} + \mu\left(\frac{1}{\mu} W_z\right)_z + k^2 n^2 W &= 0 \end{aligned} \tag{5.2}$$

where

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda} \text{ and } n^2 = \varepsilon\mu.$$

From formulas (4.8) we get expressions for E and H in terms of the new V and W; namely,

$$\begin{aligned} \varepsilon E_1 &= V_{xz} + ik\varepsilon W_y & \mu H_1 &= W_{xz} - ik\mu V_y \\ \varepsilon E_2 &= V_{yz} - ik\varepsilon W_x & \mu H_2 &= W_{yz} + ik\mu V_x \\ \varepsilon E_3 &= -V_{xx} - V_{yy} & \mu H_3 &= -W_{xx} - W_{yy} \end{aligned} \tag{5.3}$$

The continuity conditions are the same as before: V, W, $\frac{1}{\varepsilon} V_z$, and $\frac{1}{\mu} W_z$ must be continuous everywhere.

Our aim is to express the functions $V(x,y,z)$ and $W(x,y,z)$ by a complex integral and also relate V and W to the source of the electromagnetic field. We note first that special solutions of equations (5.2) are obtained by

$$\begin{aligned} V(x,y,z) &= v(z,\alpha,\beta)\, e^{ik(\alpha x + \beta y)} \\ W(x,y,z) &= w(z,\alpha,\beta)\, e^{ik(\alpha x + \beta y)} \end{aligned} \tag{5.4}$$

with arbitrary parameters α and β if $v(z,\alpha,\beta)$ and if $w(z,\alpha,\beta)$ satisfy the ordinary differential equations

$$\begin{aligned} \varepsilon\left(\frac{v'}{\varepsilon}\right)' + k^2\gamma^2 v &= 0 \\ \mu\left(\frac{w'}{\mu}\right)' + k^2\gamma^2 w &= 0 \end{aligned} \tag{5.5}$$

where $\gamma^2 = \varepsilon\mu - \alpha^2 - \beta^2 = \varepsilon\mu - \rho^2$ and primes denote differentiation with respect to z.

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12. All the following considerations are valid also in case of conducting media($\sigma \neq 0$). The function ε has only to be replaced by the complex function $\varepsilon^* = \varepsilon - \frac{4\pi\sigma}{\omega} i$.

If we now think of $\frac{\alpha}{n}$ and $\frac{\beta}{n}$ as the direction cosines of a ray with respect to the x- and y-axes, then, with some justification we may call these special solutions plane waves. Indeed by solving the second-order equations (5.5) in any region where ε and μ are constant the solutions have the form

$$e^{ik(\alpha x + \beta y \pm \sqrt{n^2 - \alpha^2 - \beta^2}\, z)}$$

which, for $\alpha^2 + \beta^2 \leq n^2$, are ordinary progressing plane waves. However, we do not exclude the case $\alpha^2 + \beta^2 > n^2$, in the following, and refer to these solutions also as plane waves.

Because equations (5.2) are linear, more general solutions can be found by superposition, namely, by integrals of the type

$$\begin{aligned} V(x,y,z) &= \iint v(z;\alpha,\beta)\, e^{ik(\alpha x + \beta y)}\, d\alpha\, d\beta . \\ W(x,y,z) &= \iint w(z;\alpha,\beta)\, e^{ik(\alpha x + \beta y)}\, d\alpha\, d\beta , \end{aligned} \tag{5.6}$$

assuming that differentiation under the integral sign is permissible for the functions v and w, with which we shall deal. Equations (5.6) show the form in which we desire to express the solution of our propagation problem. However, they do not represent the final answer to the propagation problem for we have yet to relate V and W to the source of radiation.

We now assume that at the point $x = y = z = 0$, a source of radiation is located. Some knowledge of this source must, of course, be available. We suppose that the source is given as a superposition of plane waves in a homogeneous medium ε_0, μ_0. More definitely, we suppose the source given by the complex integrals[13]

$$\begin{aligned} z > 0: \quad V &= \iint F_1(\alpha,\beta)\, e^{ik(\alpha x + \beta y + z\sqrt{n_0^2 - \rho^2})}\, d\alpha\, d\beta \\ z < 0: \quad V &= \iint F_2(\alpha,\beta)\, e^{ik(\alpha x + \beta y - z\sqrt{n_0^2 - \rho^2})}\, d\alpha\, d\beta \\ z > 0: \quad W &= \iint G_1(\alpha,\beta)\, e^{ik(\alpha x + \beta y + z\sqrt{n_0^2 - \rho^2})}\, d\alpha\, d\beta \\ z < 0: \quad W &= \iint G_2(\alpha,\beta)\, e^{ik(\alpha x + \beta y - z\sqrt{n_0^2 - \rho^2})}\, d\alpha\, d\beta \end{aligned} \tag{5.7}$$

[13] The discussion of integral representations of wave functions on pp. 361-364 of Stratton, J. A.: Electromagnetic Theory, may be helpful here.

in which $n_o = \sqrt{\varepsilon_o \mu_o}$, $\rho^2 = \alpha^2 + \beta^2$, $k = \frac{2\pi}{\lambda}$, and the F_1, F_2, G_1 and G_2 represent the amplitudes with which the plane waves enter into the combination. These integrals converge if we make the agreement that the sign of $\gamma = \sqrt{n^2 - \rho^2}$ is always to be chosen such that the imaginary part of γ is positive. If γ is real, the + sign is to be chosen. The ranges of integration in formulas (5.7) are from $-\infty$ to ∞ for α and β. Generally two expressions are needed to represent a source because the source itself has to be represented mathematically by a singularity.

As an important special case we mention the dipole oscillating in direction of the z-axis. In this case

$$F_1 = F_2 = \frac{1}{2\pi} \frac{1}{\sqrt{n_o^2 - \alpha^2 - \beta^2}} \tag{5.8}$$

$$G_1 = G_2 = 0$$

The complex integrals become

$$V = \frac{e^{ikn_o r}}{ikr} \;;\quad W = 0 \tag{5.9}$$

The proof of the result (5.9) uses a basic result in propagation theory. If we write the first and second expressions of (5.7) with the values of F_1 and F_2 given by (5.8) we obtain

$$V = \iint \frac{1}{2\pi\sqrt{n_o^2 - \alpha^2 - \beta^2}} e^{ik(\alpha x + \beta y + z\sqrt{n_o^2 - \alpha^2 - \beta^2})} \, d\alpha \, d\beta \tag{5.10}$$

We change from α, β coordinates to polar coordinates, ρ, φ. Since α and β range over the entire plane we can likewise cover the entire plane by allowing φ to range from 0 to 2π while ρ ranges from 0 to ∞. Under the transformation

$$\alpha = \rho \cos\varphi$$
$$\beta = \rho \sin\varphi$$

with the change in the element of area from $d\alpha\, d\beta$ to $\rho\, d\rho\, d\varphi$, expression (5.10) becomes

$$V = \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi\sqrt{n_o^2 - \rho^2}} e^{ik(\rho\cos\varphi\, x + \rho\sin\varphi\, y + z\sqrt{n_o^2 - \rho^2})} \rho\, d\rho\, d\varphi$$

We now let $\rho = n_o \sin\psi$;

then $d\rho = n_o \cos\psi \, d\psi$

The range of integration for ψ must now be made complex to cover the range from 0 to ∞ for ρ. As ρ goes from 0 to n_0, $\sin\psi$ goes from 0 to $\frac{\pi}{2}$; as ρ goes from n_0 to ∞, ψ must range from $\frac{\pi}{2}$ to $\frac{\pi}{2} + i\infty$. With this change of variable the expression for V becomes

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{2}+i\infty} \frac{n_0}{2\pi} e^{ikn_0(x\cos\varphi\sin\psi + y\sin\varphi\sin\psi + z\cos\psi)} \sin\psi \, d\psi \, d\varphi$$

Except for minor differences in notation and conventions as to directions for measuring angles this expression is the same as the one given in Stratton[14] for $\frac{e^{ikn_0 r}}{ikr}$[14].

The question arises as to how with a given antenna system one could know the functions V and W of such a source. Theoretically the process could be this: one might know the electromagnetic field given off by the source; in particular, one might know E_3 and H_3. By using equations (4.6) functions V and W can be obtained which are not unique but whose arbitrariness will not affect the ultimate solution. Given V and W, one can invert the double Fourier transforms in expressions (5.7) to find F_1, F_2, G_1 and G_2[15].

For some antenna systems one knows the E_3 and H_3. For example, the field of an array of dipoles is known. Likewise, field expressions for a paraboloid fed by a dipole have been obtained theoretically in the past with fairly good agreement between theory and practice. For antenna systems such as horns the field expressions are also known approximately. For paraboloidal antenna systems with more complicated feeds some theoretical work has been done which gives some knowledge of the field. For example, Radiation Laboratory Report T-9 and 762-1 by R. C. Spencer, relates the illumination supplied to the aperture of a paraboloid by the feed with the field pattern produced by the entire assembly. It is also possible, in some cases where the radiation patterns of antenna systems are known only by measurement, to obtain the functions V and W. Measurement usually produces the square of the field intensity. In those cases where the polarization is vertical, say, the square root of the measured value is E_3. However, since E_3 would be known only numerically it would be necessary either to fit a function to the data and then proceed as above

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14. Stratton, loc. cit., p. 578, formula (27).

15. Compare formulas (12) and (13) on p. 363 of Stratton, loc. cit.

or one might find V and W through equations (4.6) by some numerical process and then invert formulas (5.7) to obtain F_1, F_2, G_1, and G_2.

We return now to one major problem which is to determine the electromagnetic field radiated by a general source (5.7) if it is located at x = y = z = 0 in a medium of stratified character given by $\varepsilon(z)$, $\mu(z)$; that is, we must somehow involve the source in the equations (5.6) for V and W.

6. Propagation of an Arbitrary Source Located Below a Half-Space.

As a step in the direction of a more general propagation problem and as an indication of the method of relating a source of radiation to the field propagated let us consider the problem of a source of radiation located below a half-space of arbitrary electromagnetic character.

Specifically, a source of radiation is located at the point (0,0,0). A stratified, but otherwise arbitrary, medium is supposed to exist in the half-space $z > 0$ and a homogeneous medium is supposed to exist in the region $z < 0$. The source is known to us in the form of equations (5.7). We shall consider in the following pages only the function V since the corresponding considerations for W only involves replacing ε by μ.

We shall approach this problem by using the fact that the source can be built up of plane waves and begin a treatment of the above problem with the source, for the moment, a plane wave.

We assume that ε and μ have constant values ε_0, μ_0 for $z < 0$, and that $\varepsilon \to \varepsilon_1$, $\mu \to \mu_1$ if $z \to \infty$. A plane wave

$$V(x,y,z,t) = e^{ik(\alpha x + \beta y + z\sqrt{n_0^2 - \alpha^2 - \beta^2})} e^{-i\omega t} \tag{6.1}$$

is assumed to be incident from below. The electromagnetic field established in the region $z > 0$ under the influence of this incident plane wave then can be obtained from a function

$$V(x,y,z,t) = v(z,\alpha,\beta)\, e^{ik(\alpha x + \beta y)} e^{-i\omega t} \tag{5.4}$$

where $v(z,\alpha,\beta)$ satisfies the ordinary differential equation

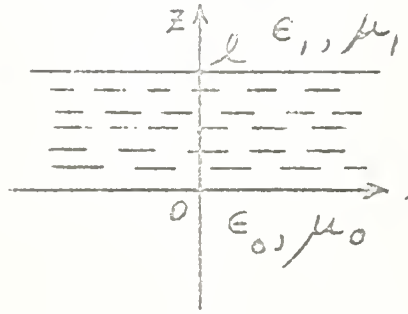
$$\varepsilon\left(\frac{v'}{\varepsilon}\right)' + k^2\gamma^2 v = 0 \tag{5.5}$$

where

$$\gamma = \sqrt{\varepsilon\mu - \alpha^2 - \beta^2} = \sqrt{n^2 - \rho^2}$$

We are led to seek a solution such as (5.4) in the region $z > 0$ because, physically, the effect of the non-homogeneous medium should be on the behavior of the wave in the z-direction. Put mathematically, we may say that if we can find a $V(x,y,z,t)$ as given by equation (5.4) where $v(z,\alpha,\beta)$ satisfies equation (5.5), we can obtain an electromagnetic field which satisfies Maxwell's equations and which can be interpreted as the alteration of the plane wave arising from the non-homogeneous layer.

We assume, at first, more specifically, that $\varepsilon = \varepsilon_1$, $\mu = \mu_1$ for $z > \ell$, the quantity ℓ being chosen arbitrarily large.



In the homogeneous parts $z < 0$ and $z > \ell$ of our medium $v(z) = v(z,\alpha,\beta)$ must have the form

$$\begin{aligned} z < 0 &: v = e^{ik\gamma_0 z} + R\, e^{-ik\gamma_0 z} \\ z > \ell &: v = T\, e^{ik\gamma_1 (z-\ell)} \end{aligned} \qquad (6.2)$$

That is, in the region $z < 0$ we allow for the presence of an incident wave of unit amplitude and a reflected wave of amplitude R, while in the medium $z > \ell$ we allow for a transmitted wave of amplitude T. The amplitudes R and T determine the <u>reflectance</u> and <u>transmittance</u> of the inhomogeneous medium with regard to the incident plane wave (6.1). The values of γ for $z < 0$ and $z > \ell$ are r_0 and r_1, respectively.

We reformulate the boundary conditions (6.2) by eliminating R and T with the aid of the derivatives $v'(o)$ and $v'(\ell)$.

We find $v(o)$ and $v'(o)$ from the first of equations (6.2) and eliminate R. We then find $v'(\ell)$ and $v(\ell)$ from the second of equations (6.2) and eliminate T. This gives

$$\begin{aligned} v'(o) + ik\gamma_0\, v(o) &= 2ik\gamma_0 \\ v'(\ell) - ik\gamma_1\, v(\ell) &= 0 \end{aligned} \qquad (6.3)$$

The plane wave problem now is to find a solution of equation (5.5) which satisfies these boundary conditions.

We can reduce this problem to the problem of finding a solution of a certain differential equation of first order of Riccati's type.

We introduce the dimensionless function

$$\Theta(z) = \frac{1}{ik}\,\frac{v'(z)}{\varepsilon v(z)} \tag{6.4}$$

Since $\frac{1}{\varepsilon} v'(z)$ and $v(z)$ are continuous functions[16] we conclude that $\Theta(z)$ must be continuous for all z even if $\varepsilon(z)$ and $\mu(z)$ should have finite discontinuities. We introduce $\frac{v'}{\varepsilon} = ik\,v\,\Theta$ in equation (5.5) and obtain an equation for Θ, namely,

$$\Theta' = ik\left(\frac{\gamma^2}{\varepsilon} - \varepsilon\Theta^2\right) \tag{6.5}$$

i.e. a simple Riccati equation. The second condition (6.3), together with equation (6.4) gives us the boundary condition for $\Theta(z)$ at $z = \ell$.

$$\Theta(\ell) = \frac{\gamma_1}{\varepsilon_1} .$$

Letting $\ell \to \infty$ we recognize $\Theta(z)$ as the solution of the Riccati equation (6.5) which satisfies the condition

$$\Theta(z) \to \frac{\gamma_1}{\varepsilon_1} \quad \text{if } z \longrightarrow \infty \tag{6.6}$$

The first condition (6.3) together with equation (6.4) can now be used to express $v(o)$ in terms of $\Theta(o)$. Also by the first equation (6.2), $v(o) = 1 + R$. These two facts together give

$$v(o) = 1 + R = \frac{2\gamma_0}{\gamma_0 + \varepsilon_0\Theta_0} , \tag{6.7}$$

where $\Theta_0 = \Theta(o)$ but ε_0 is the value of ε in the medium $z < 0$ whether or not $\varepsilon(z)$ is discontinuous at $z = 0$. The reflectance, R itself, is

$$R = \frac{\frac{\gamma_0}{\varepsilon_0} - \Theta_0}{\frac{\gamma_0}{\varepsilon_0} + \Theta_0} \tag{6.8}$$

Finally the solution, $v(z)$ itself, is found by quadratures from equation, (6.4), namely

$$\frac{v'(z)}{v(z)} = ik\,\varepsilon\,\Theta(z)$$

$$\log v(z) = \int_0^z ik\,\varepsilon\,\Theta(z)\,dz + C$$

$$v(z) = D\,e^{ik\int_0^z \varepsilon\,\Theta(z)\,dz}$$

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16. If we select a continuous solution of v for $0 \leq z \leq \ell$ of equation (5.5), which meets the boundary conditions (6.3) then $v(z)$ will be continuous everywhere. Then, by equation (5.5), $\frac{v'}{\varepsilon}$ is expressible as $\int \frac{k^2\gamma^2 v}{\varepsilon}\,dz$. Even if ε has finite discontinuities this integral is a continuous function of z for all z.

Evaluating D at z = 0 gives

$$v(z) = v(o)\ e^{ik\int_0^z \mathcal{E}\Theta(z)\ dz}$$

or, using equation (6.7),

$$v(z) = \frac{2\,\gamma_0}{\gamma_0 + \mathcal{E}_0\Theta_0}\ e^{ik\int_0^z \mathcal{E}\,\Theta(z)\,dz} \tag{6.9}$$

Returning now to equation (5.4) and omitting the time dependence we obtain the function

$$V(x,y,z) = \frac{2\,\gamma_0}{\gamma_0 + \mathcal{E}_0\,\Theta_0}\ e^{ik\left(\alpha x + \beta y + \int_0^z \mathcal{E}(z)\Theta(z)\ dz\right)} \tag{6.10}$$

This function is the solution for V of our plane wave problem in the non-homogeneous medium $z > 0$. By substituting the first of equations (6.2) in equation (5.4), with the value of R given by equation (6.8), we obtain the solution for V in the region $z < 0$.

Just to be clear on the order of events, were an actual calculation to be undertaken, one would begin by solving equation (6.5) for $\Theta(z)$ subject to the boundary condition (6.6). One then obtains $V(z)$ from equation (6.9) and then finally equation (6.10). The solution in the region $z < 0$ is then immediate from equation (6.8) and the first of equations (6.2). The theory also covers the case where ℓ is finite, in which case three functions, one each in the regions $z \leq 0$, $0 \leq z \leq \ell$, $z \geq \ell$, represent the complete solution for V. In this case one needs also the second of equations (6.2) with $T = v(\ell)$. Also the solution $\Theta(z)$ of equation (6.5) must satisfy the boundary condition $\Theta(\ell) = \dfrac{\gamma_1}{\mathcal{E}_1}$.

Since, by equation (6.1), $v(z,\alpha,\beta) = e^{ik\gamma_0 z}$ in the case of a plane wave passing through a homogeneous medium, it follows from the definition (6.4) of $\Theta(z)$ that, in an unbounded homogeneous medium, $\Theta(z) = \dfrac{\gamma_0}{\mathcal{E}_0}$ everywhere. If we now compare equations (6.10) with (6.1), to which equation (6.10) reduces when the medium is everywhere homogeneous, we see the significance of the function $\Theta(z)$. Through the expression

$$\alpha x + \beta y + \int_0^z \mathcal{E}\,\Theta(z)\ dz$$

it determines the distortion of the wave fronts $\alpha x + \beta y + \gamma z$ due to the non-homogeneous medium and through the expression

$$\frac{2\gamma_0}{\gamma_0 + \varepsilon_0 \Theta(0)}$$

it determines the change of amplitude of the undisturbed wave.

A similar result is obtained for the function $W(x,y,z)$:

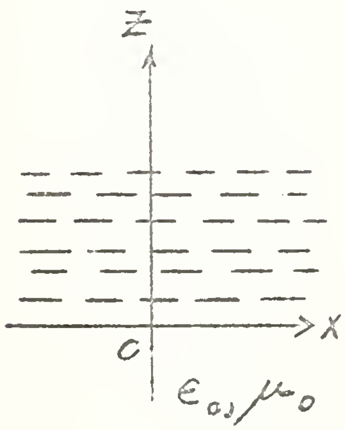
$$W(x,y,z) = \frac{2\gamma_0}{\gamma_0 + \mu_0 \theta(0)} e^{ik\left[\alpha x + \beta y + \int_0^z \mu \theta \, dz\right]} \qquad (6.11)$$

where $\theta(z)$ is the solution of the Riccati equation

$$\theta' = ik\left[\frac{\gamma^2}{\mu} - \mu\theta^2\right] \qquad (6.12)$$

which satisfies the boundary condition

$$\theta(z) \rightarrow \frac{\gamma_1}{\mu_1}, \text{ if } z \rightarrow \infty$$



6.1 The above results allow us to give immediately the formal solution of the problem of propagation with which we began article 6, namely, the propagation of an arbitrary source located below a half-space. The source of radiation is placed at the point $x = y = z = 0$. A stratified medium is considered which is homogeneous for $z < 0$. The radiation pattern of the source in a homogeneous medium with characteristics ε_0, μ_0 shall be given by integrals of the form

$$V = \iint F_1(\alpha,\beta) \, e^{ik(\alpha x + \beta y + \gamma_0 z)} \, d\alpha \, d\beta$$

$$W = \iint G_1(\alpha,\beta) \, e^{ik(\alpha x + \beta y + \gamma_0 z)} \, d\alpha \, d\beta \qquad (5.7)$$

for the region above the source.

If this source is placed below the stratified medium which begins at $z = 0$, then the associated electromagnetic field in the region $z > 0$ is determined by the two functions

$$V(x,y,z) = 2\iint \frac{\gamma_0}{\gamma_0 + \varepsilon_0 \Theta(0)} F_1(\alpha,\beta)\, e^{ik\left[\alpha x + \beta y + \int_0^z \varepsilon \Theta\, dz\right]}\, d\alpha\, d\beta$$
$$W(x,y,z) = 2\iint \frac{\gamma_0}{\gamma_0 + \mu_0 \theta(0)} G_1(\alpha,\beta)\, e^{ik\left[\alpha x + \beta y + \int_0^z \mu \theta\, dz\right]}\, d\alpha\, d\beta \qquad (6.13)$$

where γ_0, ε_0, and μ_0 are the values in the homogeneous region, and where $\Theta(z)$ and $\theta(z)$ satisfy the Riccati equations

$$\Theta'(z) = \frac{2\pi i}{\lambda}\left(\frac{\gamma^2}{\varepsilon} - \varepsilon\Theta^2\right)$$
$$\theta'(z) = \frac{2\pi i}{\lambda}\left(\frac{\gamma^2}{\mu} - \mu\theta^2\right) \qquad (6.14)$$

and boundary conditions

$$\left.\begin{aligned} \Theta(z) &\to \frac{\gamma_1}{\varepsilon_1} \\ \theta(z) &\to \frac{\gamma_1}{\mu_1} \end{aligned}\right\} \text{if } z \to \infty$$

The ranges of integration for α and β in formulas (6.13) are understood to be the same as they are for the source in formulas (5.7). The components of the electromagnetic field itself are now obtained by equations (5.3).

The justification of this general result is simple. The expressions (6.10) and (6.11) which we have obtained satisfy the requisite differential equation (5.2) because equation (6.9) and its analogue for the magnetic field satisfy equations (5.5). Expressions (6.10) and (6.11) then are the solution of the propagation problem of this article for a plane wave source. If we multiply these expressions for V and W by constant factors $F_1(\alpha,\beta)$ and $G_1(\alpha,\beta)$, respectively, for fixed α and β, we still obtain solutions of the basic equations (5.2) for V and W. Because equations (5.2) are linear, any sum of solutions finite or infinite (provided convergence criteria are satisfied) is also a solution. If, however, the source is no more than a superposition of plane waves, the propagated field of this radiation source in the non-homogeneous medium is no more than the sum of the solutions for

~~the individual plane waves. Formulas (6.13) amount to no more than the summation~~ ~~for~~ the individual plane waves. Formulas (6.13) amount to no more than the summation of an infinite number of plane waves each with its appropriate amplitude factor $F_1(\alpha, \beta)$ (or $G_1(\alpha, \beta)$), and hence must be the solution to the problem of propagating the arbitrary source (5.7) through the non-homogeneous half space.

We note that formulas (6.13) are very general for they cover any variation in ε and μ for $z \geq 0$, including finite discontinuities. The evaluation of the integrals may be a problem of some difficulty depending upon the functions F_1 and G_1 and on ε and μ. Formulas (6.13) are not, of course, the general solution of the propagation problem above a flat earth . This problem will be treated in article 7.

6.2. We shall consider one illustration of the theory just developed. As before, an arbitrary source given by expressions (5.7) is located at the point (0,0,0). The non-homogeneous medium shall consist of two layers with $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ for $z < \ell$, and $\varepsilon = \varepsilon_1$ and $\mu = \mu_1$ for $z > \ell$, $\ell > 0$. Thus our medium has a finite discontinuity in ε and μ at $z = \ell$.

In accordance with the general result of article 6.1, we must first find solutions $\Theta(z)$ and $\theta(z)$ satisfying equations (6.14) and boundary conditions (6.15). The solutions in the region where $\gamma = \gamma_1$, $\varepsilon = \varepsilon_1$, and $\mu = \mu_1$, which meet the boundary conditions (6.15) (and which therefore must be unique) are

$$z > \ell : \Theta(z) = \frac{\gamma_1}{\varepsilon_1} , \qquad \theta(z) = \frac{\gamma_1}{\mu_1} .$$

In the region $z < \ell$, $\Theta(z)$ and $\theta(z)$ must, first of all, satisfy equations (6.14), and be continuous with the solutions for $z > \ell$ at $z = \ell$. To solve equations (6.14) when γ, ε, and μ are constant we write them in the form

$$\frac{\Theta'(z)}{\left(1 - \frac{\varepsilon^2}{\gamma^2}\Theta^2(z)\right)} = \frac{2\pi i}{\lambda}\frac{\gamma^2}{\varepsilon} \quad \text{and} \quad \frac{\theta'(z)}{\left(1 - \frac{\mu^2}{\gamma^2}\theta^2(z)\right)} = \frac{2\pi i}{\lambda}\frac{\gamma^2}{\mu} .$$

which are immediately integrable by

$$\tanh^{-1} x = \int \frac{dx}{1 - x^2} .$$

To meet the continuity condition we require that

$$\Theta(\ell) = \frac{\gamma_1}{\varepsilon_1} \quad \text{and} \quad \theta(\ell) = \frac{\gamma_1}{\mu_1} .$$

Carrying out the integration and determination of the arbitrary constant by this condition, we obtain for $z < \ell$,

$$\Theta(z) = \frac{\frac{\gamma_1}{\varepsilon_1} - i\frac{\gamma_0}{\varepsilon_0}\tan\frac{2\pi}{\lambda}(\ell - z)\gamma_0}{1 - i\frac{\gamma_1}{\varepsilon_1}\frac{\varepsilon_0}{\gamma_0}\tan\frac{2\pi}{\lambda}(\ell - z)\gamma_0}, \quad \theta(z) = \frac{\frac{\gamma_1}{\mu_1} - i\frac{\gamma_0}{\mu_0}\tan\frac{2\pi}{\lambda}\gamma_0(\ell - z)}{1 - i\frac{\gamma_1}{\mu_1}\frac{\mu_0}{\gamma_0}\tan\frac{2\pi}{\lambda}\gamma_0(\ell - z)}$$

To use formula (6.13) we need also $\Theta(o)$ and $\theta(o)$, which are immediately obtainable from these last formulas.

The formulae (6.13) simplify to

$$z < \ell: \quad V = V_0 + V_1 \qquad W = W_0 + W_1$$

$$z > \ell: \quad V = V_2 \qquad W = W_2$$

where V_0 and W_0 are the given incident waves,

$$V_0 = \iint F_1(\alpha, \beta)\, e^{ik(\alpha x + \beta y + \gamma_0 z)}\, d\alpha\, d\beta$$

$$W_0 = \iint G_1(\alpha, \beta)\, e^{ik(\alpha x + \beta y + \gamma_0 z)}\, d\alpha\, d\beta$$

V_1 and W_1, the reflected waves,

$$V_1 = \iint \frac{\frac{\gamma_0}{\varepsilon_0} - \frac{\gamma_1}{\varepsilon_1}}{\frac{\gamma_0}{\varepsilon_0} + \frac{\gamma_1}{\varepsilon_1}} F_1(\alpha, \beta)\, e^{ik[\alpha x + \beta y + \gamma_0(2\ell - z)]}\, d\alpha\, d\beta$$

$$W_1 = \iint \frac{\frac{\gamma_0}{\mu_0} - \frac{\gamma_1}{\mu_1}}{\frac{\gamma_0}{\mu_0} + \frac{\gamma_1}{\mu_1}} G_1(\alpha, \beta)\, e^{ik[\alpha x + \beta y + \gamma_0(2\ell - z)]}\, d\alpha\, d\beta$$

and V_2 and W_2, the transmitted waves,

$$V_2 = \iint \frac{2\,\dfrac{\gamma_0}{\varepsilon_0}\, e^{ik\gamma_0 \ell}}{\dfrac{\gamma_0}{\varepsilon_0} + \dfrac{\gamma_1}{\varepsilon_1}}\, F_1(\alpha,\beta)\, e^{ik\left[\alpha x + \beta y + \gamma_1(z-\ell)\right]}\, d\alpha\, d\beta$$

$$W_2 = \iint \frac{2\,\dfrac{\gamma_0}{\mu_0}\, e^{ik\gamma_0 \ell}}{\dfrac{\gamma_0}{\mu_0} + \dfrac{\gamma_1}{\mu_1}}\, G_1(\alpha,\beta)\, e^{ik\left[\alpha x + \beta y + \gamma_1(z-\ell)\right]}\, d\alpha\, d\beta$$

In case $F_1(\alpha,\beta) = \dfrac{1}{2\pi\sqrt{n_0^2-\alpha^2-\beta^2}} = \dfrac{1}{2\pi\gamma_0}$

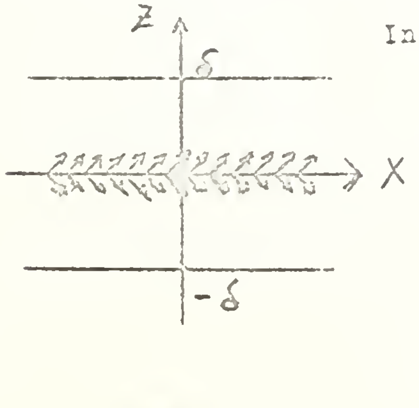
$G_1(\alpha,\beta) = 0$

we have a dipole, oscillating in the direction of the z-axis.

7. Propagation of an Arbitrary Source Located Between Two Half-Spaces.

Our major goal, namely, the propagation in a non-homogeneous atmosphere of ultra-high frequency radiation from a source located above the earth will now be approached through a more general problem. We shall suppose the source to be located at the origin of a rectangular coordinate system and allow the half-space above and below the source to have any continuous or finitely discontinuous variation of ε, μ, and σ. We may then specialize the medium below the source so that it represents the characteristics of the earth.

As in the treatment of the problem of article 6 we shall first allow the source to be a plane wave and build up the arbitrary source from plane waves. We consider then a medium which is homogeneous in a small strip $-\delta < z < \delta$ but otherwise inhomogeneous. Let us imagine that the plane $z = 0$ is the source of two plane waves symmetrically oriented to the plane $z = 0$.



In a homogeneous medium these emitted plane waves shall be

$$F_1\, e^{ik\gamma_0 z} \quad \text{for } z > 0$$

and

$$F_2\, e^{-ik\gamma_0 z} \quad \text{for } z < 0.$$

In order to determine the influence of the medium we introduce, as auxiliary quantities, two reflection coefficients R_1 and R_2. R_1 is the reflection coefficient of the upper medium $z > \delta$ if the lower medium $z < \delta$ is homogeneous. R_2 is the reflection coefficient of the lower medium if the upper medium is homogeneous.

We utilize earlier theory wherein the solution of an electromagnetic field for a plane wave through a medium varying in the z-direction only is reduced to the determination of v, as in equation (5.5), and make the substitution, as before:

$$\Theta(z) = \frac{1}{ik} \frac{v'(z)}{\varepsilon\, v(z)} \quad ,$$

which gives as before

$$\Theta' = \frac{2\pi i}{\lambda} \left(\frac{\gamma^2}{\varepsilon} - \varepsilon \Theta^2 \right) \tag{6.5}$$

The physical conditions are, of course, different from those of article 6, since we now have media of variable characteristics above and below the region $-\delta < z < \delta$ and the incident waves are in two directions. Moreover, two different solutions of equation (6.5) are needed as compared with one in article 6, for two boundary conditions now must be met, one at $z = +\infty$ and another at $z = -\infty$, and the solution of a first order equation can satisfy only one boundary condition. Let $\Theta_1(z)$ and $\Theta_2(z)$ be two solutions of equation (6.5) which shall be determined by the conditions

$$\begin{aligned} \Theta_1(z) &\to \frac{\gamma_1}{\varepsilon_1} \quad \text{if } z \to +\infty \\ \Theta_2(z) &= -\frac{\gamma_2}{\varepsilon_2} \quad \text{if } z \to -\infty; \end{aligned} \tag{7.1}$$

We may now take advantage of the theory applied in article 6 to obtain the reflection coefficient at each boundary $z = \delta$ and $z = -\delta$. As R_1 and R_2 are defined here, R_1 can be determined for the plane wave directed up and R_2 for the one directed down, without regard to the other.

Hence (cf. formula (6.8))

$$R_1 = \frac{\frac{\gamma_0}{\varepsilon_0} - \Theta_1(0)}{\frac{\gamma_0}{\varepsilon_0} + \Theta_1(0)} \quad , \qquad R_2 = \frac{\frac{\gamma_0}{\varepsilon_0} + \Theta_2(0)}{\frac{\gamma_0}{\varepsilon_0} - \Theta_2(0)} \tag{7.2}$$

Now if the medium below $z = \delta$ were homogeneous, the field for $z < \delta$ would consist of the incident field $F_1 e^{ik\gamma_0 z}$ and the reflected field $R_1 F_1 e^{ik\gamma_0 z}$. Likewise if the medium above $z = -\delta$ were homogeneous, the field arising from $F_2 e^{-ik\gamma_0 z}$ would consist of this term and $R_2 F_2 e^{-ik\gamma_0 z}$. However, the presence of both plane waves and both boundaries means that in the region $z > 0$ we shall have radiation directed towards the upper boundary which is emitted directly from the source of magnitude F_1, plus radiation reflected from the lower boundary $z = -\delta$, of ~~unknown magnitude~~ A. All of this will be partially reflected from $z = +\delta$. The reflected radiation, of amplitude B say, will be directed down. Hence there will be radiation directed towards $z = -\delta$ which consists of the source $F_2 e^{-ik\gamma_0 z}$ plus what is reflected from the layer $z = \delta$ of magnitude B. Reflected from $z = -\delta$ will be radiation of magnitude $R_2(F_2 + B)$ which in turn must be A. Likewise reflected from the upper boundary $z = \delta$ will be radiation of magnitude $R_1(F_1 + A)$ which must be B.

It follows that

$$A = \frac{R_2}{1 - R_1 R_2}(F_2 + R_1 F_1) \quad , \quad B = \frac{R_1}{1 - R_1 R_2}(F_1 + R_2 F_2) \tag{7.3}$$

From a purely mathematical standpoint we may say that in the homogeneous strip $-\delta < z < \delta$ the solution of the differential equation

$$\varepsilon \left(\frac{v'}{\varepsilon}\right)' + k^2 \gamma^2 v = 0$$

can take the form

$$\begin{aligned} v_1 &= (F_1 + A)\, e^{ik\gamma_0 z} + B\, e^{-ik\gamma_0 z}, \quad z > 0 \\ v_2 &= A\, e^{ik\gamma_0 z} + (F_2 + B)\, e^{-ik\gamma_0 z}, \quad z < 0 \end{aligned} \tag{7.4}$$

where v_1 and v_2 are two distinct solutions.

Then

$$v_1(o) = \frac{1 + R_1}{1 - R_1R_2}(F_1 + R_2F_2)$$
$$v_2(o) = \frac{1 + R_2}{1 - R_1R_2}(F_2 + R_1F_1) \tag{7.5}$$

To obtain the solutions in the non-homogeneous media we now treat $v_1(z)$ and $v_2(z)$ as in article 6. We repeat the steps formerly made to obtain equation (6.9) from equation (6.4) only the $v_1(o)$ here replaces the $v(o)$ equation (6.7). We repeat the process once more, this time using $v_2(o)$ to replace $v(o)$. We then substitute each result in equation (5.4). Now δ is arbitrarily small. Hence we may say that .

$$z < 0 : \quad V_2(x,y,z) = \frac{1 + R_1}{1 - R_1R_2}(F_1 + R_2F_2)\, e^{ik(\alpha x + \beta y + \int_0^z \varepsilon\Theta_2\, dz)}$$
$$z > 0 : \quad V_1(x,y,z) = \frac{1 + R_2}{1 - R_1R_2}(F_2 + R_1F_1)\, e^{ik(\alpha x + \beta y + \int_0^z \varepsilon\Theta_1\, dz)} \tag{7.6}$$

This is the solution of our problem of propagation for plane wave sources. If the source is symmetrical, i.e., if $F_1 = F_2 = F = 1$ then our result simplifies considerably by introducing $\Theta_1(o)$ and $\Theta_2(o)$ in (7.5) by means of (7.2). We obtain

$$z < 0 : V_2(x,y,z) = \frac{2\gamma_o/\varepsilon_o}{\Theta_1(o) - \Theta_2(o)}\, e^{ik(\alpha x + \beta y + \int_0^z \varepsilon\Theta_2\, dz)}$$
$$z > 0 : V_1(x,y,z) = \frac{2\gamma_o/\varepsilon_o}{\Theta_1(o) - \Theta_2(o)}\, e^{ik(\alpha x + \beta y + \int_0^z \varepsilon\Theta_1\, dz)} \tag{7.7}$$

Thus if the source consists of two plane waves, one of amplitude F_1 directed upwards and one of amplitude F_2 directed downward, then the effect of the medium on the wave is given by formulas (7.6) with corresponding expressions for W. If the two plane waves have amplitude 1 then the effect of the medium is given by expressions (7.7) with corresponding expressions for W. The full field expressions for plane waves in the non-homogeneous media are then obtained by means of relations (5.3).

We turn now to the major problem of a general source of radiation placed at $x = y = z = 0$ with the media above and below non-homogeneous to the extent of being stratified. For this more general source we reason again as in article 6. The character of the source must be known to us either through the functions $F(\alpha, \beta)$ and $G(\alpha, \beta)$ or through V and W for a homogeneous medium from which we may find F and G through relations (5.7) by inverting the Fourier transforms.

Again we argue that since our general source is a superposition of plane waves each modified by the amplitude factor $F(\alpha, \beta)$, the superposition of the plane waves modified by the presence of the non-homogeneous medium and each multiplied by its amplitude factor $F(\alpha, \beta)$ must be a solution of Maxwell's equations, for these equations, or the equivalent formulation (5.2) and (5.3) of this paper, are linear.

Specifically, let us assume that the source is symmetrical so that $F_1 = F_2 = F(\alpha, \beta)$ and $G_1 = G_2 = G(\alpha, \beta)$ in formula (5.7). Then we obtain the functions $V(x,y,z)$ and $W(x,y,z)$ by superposing "plane" waves of the form (7.7). We obtain

$$z > 0: \begin{cases} V_1(x,y,z) = \dfrac{2}{\varepsilon(o+)} \displaystyle\iint \dfrac{\gamma(o+)}{\Theta_1(o) - \Theta_2(o)} F(\alpha, \beta) e^{ik(\alpha x + \beta y + \int_0^z \varepsilon \Theta_1 dz)} d\alpha d\beta \\ W_1(x,y,z) = \dfrac{2}{\mu(o+)} \displaystyle\iint \dfrac{\gamma(o+)}{\theta_1(o) - \theta_2(o)} G(\alpha, \beta) e^{ik(\alpha x + \beta y + \int_0^z \mu \theta_1 dz)} d\alpha d\beta \end{cases} \qquad (7.8)$$

and

$$z < 0: \begin{cases} V_2(x,y,z) = \dfrac{2}{\varepsilon(o-)} \displaystyle\iint \dfrac{\gamma(o-)}{\Theta_1(o) - \Theta_2(o)} F(\alpha, \beta) e^{ik(\alpha x + \beta y + \int_0^z \varepsilon \Theta_2 dz)} d\alpha d\beta \\ W_2(x,y,z) = \dfrac{2}{\mu(o-)} \displaystyle\iint \dfrac{\gamma(o-)}{\theta_1(o) - \theta_2(o)} G(\alpha, \beta) e^{ik(\alpha x + \beta y + \int_0^z \mu \theta_2 dz)} d\alpha d\beta \end{cases} \qquad (7.9)$$

where $\gamma(o+)$, $\mu(o+)$, and $\varepsilon(o+)$ are the values approached by γ, μ, and ε as z approaches o from above, and $\gamma(o-)$, $\mu(o-)$, and $\varepsilon(o-)$ are the values of γ, μ, and ε as z approaches o from below; and where $\Theta_1(z)$, $\Theta_2(z)$ and $\theta_1(z)$, $\theta_2(z)$ are the solutions of the Riccati equations

$$\Theta' = \frac{2\pi i}{\lambda} \left(\frac{\gamma^2}{\varepsilon} - \varepsilon \Theta^2\right) \qquad (7.10)$$

$$\theta' = \frac{2\pi i}{\lambda} \left(\frac{\gamma^2}{\mu} - \mu \theta^2\right)$$

which satisfy the boundary conditions

$$\left.\begin{aligned} \Theta_1(z) &\to \frac{\gamma_1}{\varepsilon_1} \\ \theta_1(z) &\to \frac{\gamma_1}{\mu_1} \end{aligned}\right\} \quad \text{if } z \to +\infty \tag{7.11}$$

and

$$\left.\begin{aligned} \Theta_2(z) &\to -\frac{\gamma_2}{\varepsilon_2} \\ \theta_2(z) &\to -\frac{\gamma_2}{\mu_2} \end{aligned}\right\} \quad \text{if } z \to -\infty$$

It is understood that here, too, the ranges of integration for α and β are determined by the source.

The results embodied in formulas (7.8) and (7.9) taken in conjunction with formulas (5.3) constitute the major general conclusion of this paper. They solve the problem of propagation in stratified non-homogeneous media in the following sense. One starts with the nature of the source, that is, one either knows the functions $F(\alpha, \beta)$ and $G(\alpha, \beta)$ or, alternatively, the functions V and W of relations (5.7) from which F and G can be obtained. Physically this means that one knows how the source can be built up from plane waves. Granted this, formulas (7.8) and (7.9) are used to calculate V and W of the modified field and then the field components can be found by means of formulas (5.3).

The nature of the inhomogeneity of the medium above and below the source is, of course, implicit in the functions Θ_1, θ_1, Θ_2 and θ_2 which are determined first by solutions of equations (7.10) with the boundary conditions (7.11). The character of the inhomogeneity is restricted only to the extent that ε, μ, and σ must be functions (continuous or with a finite number of finite discontinuities) of z. It will be recalled also that if the medium has conductivity, ε must be replaced by $\varepsilon - \frac{4\pi\sigma i}{\omega}$. Solution of equations (7.10), which are of the Riccati type, may have to be approximated. The ease or difficulty with which one obtains solutions depends upon the nature of the functions ε and μ.

It will be observed that the result for V and W in (7.8) and (7.9) are complex integrals. The evaluation of these integrals is, in general not simple, even though exact or approximate expressions are known for Θ_1, Θ_2, θ_1 and θ_2, for in the integrals the nature of the $F(\alpha, \beta)$ and $G(\alpha, \beta)$ also must be considered.

Articles 7.1 to 7.4 will take up special cases of this general result. These special cases not only will give some indication of the difficulties met in evaluation but will come closer to more practical situations.

7.1 <u>The Dipole Source between Non-Homogeneous Stratified Media.</u>

The radiation of a dipole oscillating in the direction of the z-axis, is obtained by letting $F(\alpha, \beta) = \frac{1}{2\pi \gamma_0}$ and $G(\alpha, \beta) = 0$. (See equation (5.8) above.) Formulas (7.8) and (7.9) thus yield for, $z > 0$,

$$V_1 = \frac{1}{2\pi\, \varepsilon(0)} \iint \frac{e^{ik(\alpha x + \beta y + \int_0^z \varepsilon\, \Theta_1\, dz)}}{\Theta_1(0) - \Theta_2(0)}\, d\alpha\, d\beta \qquad (7.12)$$

$$W_1 = 0.$$

and for $z < 0$,

$$V_2 = \frac{1}{2\pi\, \varepsilon(0)} \iint \frac{e^{ik(\alpha x + \beta y + \int_0^z \varepsilon\, \Theta_2\, dz)}}{\Theta_1(0) - \Theta_2(0)}\, d\alpha\, d\beta \qquad (7.13)$$

$$W_2 = 0$$

As indicated in the discussion following equation (6.10), for the unbounded homogeneous medium, $\Theta_1 = \frac{\gamma_0}{\varepsilon_0}$, $\Theta_2 = -\frac{\gamma_0}{\varepsilon_0}$,

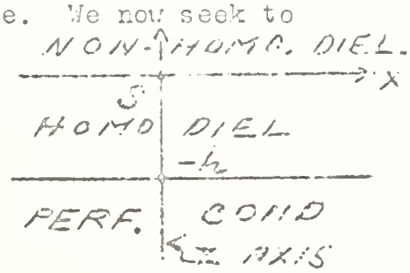
and we find

$$V = \frac{e^{ikn_0 r}}{ikr}$$

as it should be (See the proof of formula (5.9))

7.2 <u>A General Source Above a Perfectly Conducting Ground Surface.</u>

The general theory of article 7 allows for any type of (stratified) non-homogeneous medium above and below the arbitrary source. We now seek to specialize this to the case where there is a homogeneous layer of height h below the source and below this a perfect reflector, which, in practical problems, would be the earth. The source itself is assumed to be at the origin (0,0,0).



To obtain solutions for V and W for this problem we shall need expressions in the three different regions $z > 0$, $-h < z < 0$, and $z < -h$. The third case may be neglected, for no field exists in a perfect conductor. The region $z > 0$ is still unspecialized beyond the fact that it is stratified. We can be specific within the homogeneous layer. We must first solve equations (7.10) for the region $-h < z < 0$ subject to the following conditions. Within this region ε, μ, and γ are constant and denoted by ε_0 μ_0, and γ_0.

We note that within the perfect conductor the dielectric constant $\varepsilon - \frac{4\pi\sigma i}{\omega}$ is infinite. This occurs at $z = -h$. Since $\gamma^2 = \varepsilon\mu - \alpha^2 - \beta^2$ .. whatever the value of α and β, our boundary conditions, which replace conditions (7.11), are

$$\Theta_2(-h) = 0$$

$$\theta_2(-h) = \infty$$

The solutions[17] of equation (7.10) which hold in $-h < z < 0$ and satisfy these conditions are:

$$\Theta_2(z) = i\,\frac{\gamma_0}{\varepsilon_0}\,\tan\,\frac{2\pi}{\lambda}\gamma_0(z+h)$$
$$\theta_2(z) = i\,\frac{\gamma_0}{\mu_0}\,\cot\,\frac{2\pi}{\lambda}\gamma_0(z+h) \qquad (7.14)$$

In particular

$$\Theta_2(o) = i\,\frac{\gamma_0}{\varepsilon_0}\,\tan\,\frac{2\pi}{\lambda}\gamma_0 h$$

$$\theta_2(o) = i\,\frac{\gamma_0}{\mu_0}\,\cot\,\frac{2\pi}{\lambda}\gamma_0 h$$

These four quantities $\Theta_2(z)$, $\theta_2(z)$, $\Theta_2(o)$ and $\theta_2(o)$ must now be substituted in formulas (7.8) and (7.9) to obtain the field above and below the source. One cannot proceed to evaluate the double integrals because thus far the character of the medium above the source has been left quite general; that is, the four quantities $\Theta_1(z)$, $\theta_1(z)$, $\Theta_1(o)$, and $\theta_1(o)$ are undetermined. (Particular media above the dipole will be considered in later articles.

The theory of this article calls for a homogeneous layer below the source. The parameters ε, μ, and σ in this layer can be any constants followed by an arbitrary variation in these parameters in the non-homogeneous medium above the source. The restriction to a homogeneous layer below the source is merely for purposes of illustration. The general theory of article 7 permits any arbitrary medium

17. The method of solution is the same as in article 6.2.

between the source and the perfectly conducting earth. If, as in the present article, the layer between the source and the perfect earth extends from $-h \leq z \leq 0$, the boundary conditions $\Theta_2(-h)$ and $\theta_2(-h) = \infty$ would be retained, but equations (7.14) would not be used. They must be replaced by more general solutions of equations (7.10) wherein $\mathcal{E}$ and μ and have functional forms determined by the non-homogeneity in the layer.

Of course, all of the theory of this article applies when the source is a dipole and, in fact, simplifies considerably as article 7.1 illustrates.

If $h = 0$ or if $\frac{h}{\lambda} << 1$ we have

$$\Theta_2(o) = 0$$

$$\theta_2(o) = \infty$$

and thus obtain from equation (7.8) the simplified expressions

$$V(x,y,z) = \frac{2}{\mathcal{E}(o)} \iint \frac{\gamma(o)}{\Theta_1(o)} F(\alpha,\beta)\, e^{ik(\alpha x + \beta y + \int_0^z \mathcal{E}\, \Theta_1 dz)}\, d\alpha\, d\beta \quad (7.15)$$

$$W(x,y,z) = 0$$

Thus equation (7.15) with equations (5.3) gives the field of an arbitrary source situated <u>on</u> a perfectly conducting (flat) earth above which there is a nonhomogeneous stratified medium.

7.3 <u>The Special Case of a Multilayer.</u>

In the event that either or both stratified nonhomogeneous media consists of a number of homogeneous layers separated by parallel planes the above theory applies since the discontinuities in $\mathcal{E}$, μ, and σ are of the finite jump type. We shall apply the basic theory of article 7 to this case.

To determine the field which arises from an arbitrary source we must, in accordance with equations (7.8) and (7.9), determine first the $\Theta(z)$ and $\theta(z)$ above and below the source. Let us suppose that the layers so that the effect of the layers will be on the form of $\Theta_1(z)$ and $\theta_1(z)$ in the notation and theory of article 7.

In accordance with our general theory we must first solve the Riccati equation (7.10), namely,

$$\Theta' = \frac{2\pi i}{\lambda}\left(\frac{\gamma^2}{\varepsilon} - \varepsilon\theta^2\right)$$

in the region $z > 0$. Since ε, μ, and σ may have finite discontinuities as we pass from one layer to another it is necessary to obtain a separate solution for Θ in each layer, though as remarked in article 6, $\Theta(z)$ will be continuous throughout the multilayer.

Let ε_ν, μ_ν be the material constants of the ν-th layer and let ℓ_ν be its thickness. Since ε_ν and γ_ν are constant within the ν-th layer we can solve for Θ directly, as follows. We first restate equation (7.10) thus:

$$\frac{\dfrac{\varepsilon_\nu}{\gamma_\nu}\Theta'}{1 - \dfrac{\varepsilon_\nu^2}{\gamma_\nu^2}\Theta^2} = \frac{2\pi i}{\lambda}\gamma_\nu$$

$$\frac{1}{2}\cdot\frac{d}{dz}\left(\log\frac{1 + \dfrac{\varepsilon_\nu}{\gamma_\nu}\Theta}{1 - \dfrac{\varepsilon_\nu}{\gamma_\nu}\Theta}\right) = \frac{2\pi i}{\lambda}\gamma_\nu$$

$$\frac{1 + \dfrac{\varepsilon_\nu}{\gamma_\nu}\Theta}{1 - \dfrac{\varepsilon_\nu}{\gamma_\nu}\Theta} = Ce^{\frac{\mp 2\pi i}{\lambda}\gamma_\nu z} \tag{7.16}$$

where C is a constant of integration[18]. If we compute the definite integral from Θ_ν to $\Theta_{\nu+1}$, the values of Θ at the bottom and top of the ν-th layer, we obtain

$$\frac{1 + \dfrac{\varepsilon_\nu}{\gamma_\nu}\Theta_{\nu+1}}{1 - \dfrac{\varepsilon_\nu}{\gamma_\nu}\Theta_{\nu+1}} = \frac{1 + \dfrac{\varepsilon_\nu}{\gamma_\nu}\Theta_\nu}{1 - \dfrac{\varepsilon_\nu}{\gamma_\nu}\Theta_\nu}\, e^{\frac{\mp 2\pi i}{\lambda}\gamma_\nu \ell_\nu} \tag{7.17}$$

---

18. C is not arbitrary. We do not require its value in the steps immediately following but it can be determined by the condition that $\Theta(z)$ in the ν-th layer must be continuous with $\Theta(z)$ in the $(\nu+1)$-st layer. Finally, as the next few steps indicate, $\Theta(z)$ in the uppermost layer is constant and determined.

By solving for Θ_ν and using trigonometric identities, we obtain

$$\Theta_\nu = \frac{\Theta_{\nu+1} - i\,\frac{\gamma_\nu}{\varepsilon_\nu}\tan\frac{2\pi}{\lambda}\gamma_\nu \ell_\nu}{1 - i\,\frac{\varepsilon_\nu}{\gamma_\nu}\Theta_{\nu+1}\tan\frac{2\pi}{\lambda}\gamma_\nu \ell_\nu} \tag{7.18}$$

which allows us to determine successively the quantities Θ_ν from the last one

$$\Theta_m = \frac{\gamma_m}{\varepsilon_m} .$$

which is the solution of the basic equation (7.10) for Θ in the semi-infinite region $z > \ell_1 + \ell_2 + \quad + \ell_m$, wherein γ and ε have the values γ_m and ε_m.

A similar formula is obtained for θ_ν, namely

$$\theta_\nu = \frac{\theta_{\nu+1} - i\,\frac{\gamma_\nu}{\mu_\nu}\tan\frac{2\pi}{\lambda}\gamma_\nu \ell_\nu}{1 - i\,\frac{\mu_\nu}{\gamma_\nu}\theta_{\nu+1}\tan\frac{2\pi}{\lambda}\gamma_\nu \lambda_\nu} \tag{7.19}$$

$$\theta_m = \frac{\gamma_m}{\mu_m}$$

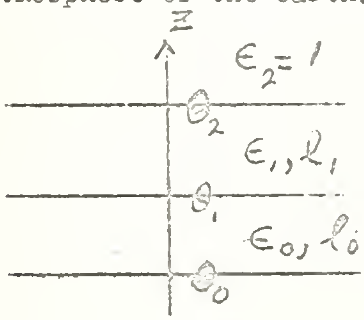
We can now use these formulas (7.18) and (7.19) to obtain some of the quantities required by the general formulas (7.8) and (7.9) or by any specialization of these formulas. These general formulas call for the values $\Theta_1(o)$ and $\theta_1(o)$. In our case, where the multilayer lies immediately above the source, $\Theta_1(o) = \Theta_o$ and $\theta_1(o) = \theta_o$. The quantities Θ_o and θ_o can be obtained by computation with the recursion formulas (7.18) and (7.19).

The general formulas (7.8) and (7.9) require also the functional expressions $\Theta_1(z)$ and $\theta_1(z)$, $\Theta_2(z)$ and $\theta_2(z)$. In our case, where the multilayer lies immediately above the source, $\Theta_2(z)$ and $\theta_2(z)$ are unspecialized and depend upon the medium below. $\Theta_1(z)$ and $\theta_1(z)$, however, depend upon the multilayer and can be obtained from formula (7.16) and the corresponding formula wherein μ_ν replaces ε_ν (see article 7.4). We note that the expressions for $\Theta_1(z)$ and $\theta_1(z)$ will vary from layer to layer because ε_ν, μ_ν, and γ_ν change. As a consequence the expressions

for V and W given by formulas (7.8) and (7.9) will change from layer to layer, and ultimately so will the expressions for the components of E and H as determined by equations (5.3).

7.4 The Dipole in a Multilayer of Three Strata

The formulas of article 7.3 allow us in principle to express the solution of the case of the multilayer by a series of integrals. However, the integrals become complicated rapidly if the number of layers is large. We treat in the following a simple example, which represents roughly to a certain extent the situation in the atmosphere of the earth. Consider a medium of three strata



a) $0 \leqq z < l_0$; ε_0 fixed and $\mu_0 = 1$

b) $l_0 \leqq z \leqq l_0 + l_1$; ε_1 fixed and $\mu_1 = 1$

c) $l_0 + l_1 \leqq z$; $\varepsilon_2 = \mu_2 = 1$

The ground surface is assumed to be perfectly conducting and we shall therefore use results of article 7.2. The source shall be a dipole oscillating in the direction of the z-axis and located at (o,o,o) so that, in view of formulas (5.7) and (5.8),

$$V = \frac{e^{ik\sqrt{\varepsilon_0}\, r}}{ikr} = \frac{1}{2\pi}\iint e^{ik(\alpha x + \beta y + z\sqrt{\varepsilon_0 - \alpha^2 - \beta^2})} \frac{d\alpha\, d\beta}{\sqrt{\varepsilon_0 - \alpha^2 - \beta^2}}$$

$$W = 0$$

if the medium in which the dipole lies is homogeneous with $\varepsilon = \varepsilon_0$.

By means of formula (7.15) we obtain the solution for the above inhomogeneous medium.

Since $\gamma_0 = \sqrt{\varepsilon_0 \mu_0 - \alpha^2 - \beta^2}$, the expression for V becomes

$$V(x,y,z) = \frac{1}{\pi\, \varepsilon_0}\iint \frac{1}{\Theta_0}\, e^{ik(\alpha x + \beta y + \int_0^z \varepsilon\, \Theta(z)\, dz)}\, d\alpha\, d\beta \tag{7.20}$$

The function $\Theta(z)$ will be found by the aid of the formula (7.16).

Before doing this let us consider the special case of a completely homogeneous medium $\mathcal{E} = \mathcal{E}_0$ above the conducting ground surface. We have

$$\Theta(z) = \frac{1}{\mathcal{E}_0} \gamma_0 = \frac{1}{\mathcal{E}_0} \sqrt{\mathcal{E}_0 - \alpha^2 - \beta^2}$$

and hence

$$V(x,y,z) = \frac{1}{\pi} \iint e^{ik(\alpha x + \beta y + z\sqrt{\mathcal{E}_0 - \alpha^2 - \beta^2})} \frac{d\alpha\, d\beta}{\sqrt{\mathcal{E}_0 - \alpha^2 - \beta^2}}$$

$$= 2 \frac{e^{ik\sqrt{\mathcal{E}_0}\, r}}{ikr}$$

i.e. the original source with four times the intensity $(|V|^2)$.

We return to the case of the three layers. Since within any one layer $\mathcal{E}$ is constant, $\Theta(z)$ no longer depends upon $\mathcal{E}$ but only on $\rho = \sqrt{\alpha^2 + \beta^2}$.

Formula (7.20) may then be rewritten and transformed as follows: We have directly

$$V(x,y,z) = \frac{1}{\pi \mathcal{E}_0} \iint \frac{1}{\Theta_0} e^{ik(\alpha x + \beta y)} e^{ik\int_0^z \mathcal{E}\, \Theta(z)\, dz}\, d\alpha\, d\beta$$

We now make the following changes of variables.

Let

$$\alpha = \rho \cos \varphi$$
$$\beta = \rho \sin \varphi$$
$$x = r \cos \psi$$
$$y = r \sin \psi$$

The change from α, β to ρ, φ involves a change in the variables of integration as was performed in article 5. The change from x, y to r, ψ does not involve the variables of integration.

We obtain

$$V(x,y,z) = \frac{1}{\pi \mathcal{E}_0} \int_0^{2\pi}\!\!\int_0^{\infty} \frac{1}{\Theta_0} e^{ikr\rho\cos(\varphi - \psi)} e^{ik\int_0^z \mathcal{E}\, \Theta(z)\, dz} \rho\, d\rho\, d\varphi$$

We now use the fact that $\Theta(z)$ depends only upon ρ to write our integral thus:

$$V(x,y,z) = \frac{1}{\pi \epsilon_o} \int_0^{\infty} \left[\frac{1}{\Theta_o(\rho)} e^{ik\int_0^z \epsilon \Theta(z)\,dz} \int_{-\pi}^{\pi} e^{ikr\rho \cos n}\,dn \right] \rho\, d\rho$$

wherein $n = (\varphi - \psi)$ and we have changed the bounds of integration of n to $-\pi$ to π, which we may do because the integrand is periodic.

We now use a standard integral expression for $J_o(x)$, namely,[19]

$$J_o(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix \cos \psi}\, d\psi$$

to rewrite our integral as

$$V(x,y,z) = \frac{2}{\epsilon_o} \int_0^{\infty} \frac{1}{\Theta_o(\rho)} J_o(k\rho r)\, e^{ik\int_0^z \epsilon \Theta(z)\,dz} \rho\, d\rho \tag{7.21}$$

To evaluate V we must first determine $\Theta_o(\rho)$ and the value of the $\int \epsilon \Theta(z)\, dz$. We proceed to do this. For the determination of Θ_o we have, by the recursion formulas (7.18),

$$\Theta_2 = \frac{\gamma_2}{\epsilon_2} = \sqrt{1 - \rho^2} = s$$

$$\Theta_1 = \frac{s - \frac{i}{\epsilon_1} \gamma_1 \tan \frac{2\pi}{\lambda} \gamma_1 \ell_1}{1 - i \frac{\epsilon_1}{\gamma_1} s \tan \frac{2\pi}{\lambda} \gamma_1 \ell_1} \tag{7.22}$$

$$\Theta_o = \frac{\Theta_1 - i \frac{\gamma_o}{\epsilon_o} \tan \frac{2\pi}{\lambda} \gamma_o \ell_o}{1 - i \frac{\epsilon_o}{\gamma_o} \Theta_1 \tan \frac{2\pi}{\lambda} \gamma_o \ell_o}$$

Since $\gamma_1 = \sqrt{\epsilon_1 - \rho^2} = \sqrt{\epsilon_1 - 1 + s^2}$

$\gamma_o = \sqrt{\epsilon_o - \rho^2} = \sqrt{\epsilon_o - 1 + s^2}$

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19. Stratton, loc. cit., p. 367.

and since only the even functions of either radical, namely,

$$\gamma \tan \frac{2\pi}{\lambda} \gamma \ell$$

$$\frac{1}{\gamma} \tan \frac{2\pi}{\lambda} \gamma \ell$$

enter into formulas (7.22), the value of $\Theta_o(s)$ for the positive or negative value of the radical must be the same. Hence Θ_o is a one-valued function of s, i.e. the Riemann surface of $\Theta_o(s)$ has only one sheet. The only possible singularities of the function $\frac{1}{\Theta_o(s)}$ thus are poles. Their location is given by the roots of the equation $\Theta_o(s) = 0$, i.e. by

$$i \frac{\gamma_o}{\varepsilon_o} \tan \frac{2\pi}{\lambda} \gamma_o \ell_o = \frac{s - \frac{i}{\varepsilon_1} \gamma_1 \tan \frac{2\pi}{\lambda} \gamma_1 \ell_1}{1 - i \frac{\varepsilon_1}{\gamma_1} s \tan \frac{2\pi}{\lambda} \gamma_1 \ell_1} .$$

We next determine the function

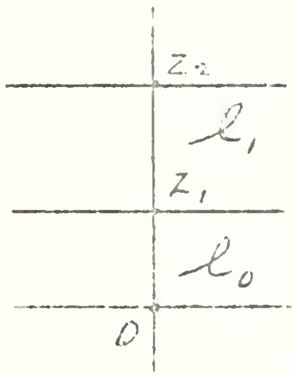
$$v(z) = \frac{1}{\Theta_o} e^{ik\int_0^z \varepsilon \Theta(z)\, dz} \tag{7.23}$$

in formula (7.21). We can do this by first solving the Riccati equation (7.10) for $\Theta(z)$ and then carrying out the integration. However, it is simpler to proceed as follows. We know by formula (5.5) that v(z) must be a solution of the differential equation

$$\varepsilon\left(\frac{v'}{\varepsilon}\right)' + k^2 (\varepsilon - \rho^2)\, v = 0$$

Since $\varepsilon(z)$ is constant in the individual layers, we may treat this differential equation as an ordinary second-order equation, and we conclude that v(z) must have the form

$$v(z) \begin{cases} = A_o e^{ik\gamma_o z} + B_o e^{-ik\gamma_o z}, & 0 \leqq z < z_1 = \ell_o \\ = A_1 e^{ik\gamma_1(z-z_1)} + B_1 e^{-ik\gamma_1(z-z_1)}, & \ell_o \leqq z \leqq z_2 = \ell_o + \ell_1 \\ = A_2 e^{ik\gamma_2(z-z_2)}, & z_2 \leqq z . \end{cases} \tag{7.24}$$



The special form of $v(z)$ in the medium $z > z_2$ is a consequence of the boundary condition that $\Theta(z) \to \gamma/\varepsilon$ as $z \to \infty$. Since $\Theta(z) = \frac{1}{ik}\frac{v'(z)}{\varepsilon v(z)}$ this condition is $\frac{v'}{\varepsilon} - ik\frac{\gamma}{\varepsilon} v \to 0$ if $z \to \infty$. From equation (7.23) it follows that

$$v'(o) = ik\varepsilon_o ;$$

and hence, from the first of expressions (7.24),

$$A_o - B_o = \frac{\varepsilon_o}{\gamma_o} \tag{7.25}$$

We now impose the conditions (see statement (5.1)) that $v(z)$ and $\frac{1}{\varepsilon} v'(z)$ must be continuous everywhere. These conditions require that the first two expressions for $v(z)$ and $v'(z)$ in (7.24) must agree at $z_1 = \ell_o$, and the latter two expressions and their derivatives must agree at $z_2 = \ell_o + \ell_1$. This gives us a system of four linear equations for the quantities A_o, B_o, A_1, B_1, and A_2, namely,

$$\begin{aligned}
A_o e^{ik\gamma_o \ell_o} + B_o e^{-ik\gamma_o \ell_o} &= A_1 + B_1 \\
A_o e^{ik\gamma_o \ell_o} - B_o e^{-ik\gamma_o \ell_o} &= \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}(A_1 - B_1) \\
A_1 e^{ik\gamma_1 \ell_1} + B_1 e^{-ik\gamma_1 \ell_1} &= A_2 \\
A_1 e^{ik\gamma_1 \ell_1} - B_1 e^{-ik\gamma_1 \ell_1} &= \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1} A_2
\end{aligned} \tag{7.26}$$

or

$$\begin{aligned}
2A_o &= e^{-ik\gamma_o \ell_o}\left\{\left(1 + \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}\right) A_1 + \left(1 - \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}\right) B_1\right\} \\
2B_o &= e^{ik\gamma_o \ell_o}\left\{\left(1 - \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}\right) A_1 + \left(1 + \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}\right) B_1\right\} \\
2A_1 &= \left(1 + \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1}\right) A_2 e^{-ik\gamma_1 \ell_1} \\
2B_1 &= \left(1 - \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1}\right) A_2 e^{ik\gamma_1 \ell_1}
\end{aligned}$$

These equations allow us to express the quantities A_o, B_o, A_1, and B_1 in terms of quantity A_2 :

$$A_o = \frac{1}{4} A_2 e^{-ik\gamma_o \ell_o} \left\{ \left(1 + \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}\right)\left(1 + \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1}\right) e^{-ik\gamma_1 \ell_1} + \left(1 - \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}\right)\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1}\right) e^{ik\gamma_1 \ell_1} \right\}$$

$$B_o = \frac{1}{4} A_2 e^{ik\gamma_o \ell_o} \left\{ \left(1 - \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}\right)\left(1 + \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1}\right) e^{-ik\gamma_1 \ell_1} + \left(1 + \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}\right)\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1}\right) e^{ik\gamma_1 \ell_1} \right\} \quad (7.27)$$

$$A_1 = \frac{1}{2} A_2 e^{-ik\gamma_1 \ell_1} \left(1 + \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1}\right)$$

$$B_1 = \frac{1}{2} A_2 e^{ik\gamma_1 \ell_1} \left(1 - \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1}\right)$$

By subtracting the second equation from the first, replacing $A_o - B_o$ by $\frac{\varepsilon_o}{\gamma_o}$ in accordance with formula (7.25) and solving for A_2 we obtain

$$A_2 = \frac{1}{\Omega}$$

where Ω is defined by the equation

$$\begin{aligned} 4\frac{\varepsilon_o}{\gamma_o}\Omega = {} & \left(1 + \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}\right)\left(1 + \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1}\right) e^{-ik(\gamma_o \ell_o + \gamma_1 \ell_1)} \\ & + \left(1 - \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}\right)\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1}\right) e^{ik(\gamma_1 \ell_1 - \gamma_o \ell_o)} \\ & - \left(1 - \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}\right)\left(1 + \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1}\right) e^{ik(\gamma_o \ell_o - \gamma_1 \ell_1)} \\ & - \left(1 + \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o}\right)\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\frac{\gamma_2}{\gamma_1}\right) e^{ik(\gamma_o \ell_o + \gamma_1 \ell_1)} \end{aligned} \quad (7.28)$$

By using the familiar relations between trigonometric and exponential functions one verifies readily that Ω can be written as follows, letting

$$\gamma_2 = \sqrt{1-\rho^2} = s , \quad \varepsilon_2 = 1 ;$$

$$\Omega = s \left[\cos k\gamma_o \ell_o \cos k\gamma_1 \ell_1 + \frac{\gamma_o \varepsilon_1}{\gamma_1 \varepsilon_o} \sin k \gamma_o \ell_o \sin k \gamma_1 \ell_1 \right] \tag{7.29}$$

$$-i \left[\frac{\gamma_o}{\varepsilon_o} \sin k\gamma_o \ell_o \cos k\gamma_1 \ell_1 + \frac{\gamma_1}{\varepsilon_1} \cos k\gamma_o \ell_o \sin k\gamma_1 \ell_1 \right]$$

which shows that <u>Ω(s) is an integral function of s, i.e. regular in the whole s-plane.</u> With the aid of this result we may finally return to formulas (7.24) and express the function v(z) itself in the following explicit form:

<u>for $0 < z < z_1 = \ell_o$:</u>

$$v(z) = \frac{1}{\Omega(s)} \Big\{ \cos k\gamma_o(z-z_1) \cos k\gamma_1 \ell_1 + \frac{\varepsilon_o}{\varepsilon_1}\frac{\gamma_1}{\gamma_o} \sin k\gamma_o(z-z_1) \sin k\gamma_1 \ell_1$$

$$- is \left[\frac{\varepsilon_1}{\gamma_1} \sin k\gamma_1 \ell_1 \cos k\gamma_o(z-z_1) - \frac{\varepsilon_o}{\gamma_o} \cos k\gamma_1 \ell_1 \sin k\gamma_o(z-z_1) \right] \Big\} \tag{7.30}$$

<u>for $z_1 < z < z_2$</u>

$$v(z) = \frac{1}{\Omega(s)} \quad \cos k\gamma_1(z-z_2) + i s \frac{\varepsilon_1}{\gamma_1} \sin k \gamma_1(z-z_2)$$

<u>for $z > z_2$</u>

$$v(z) = \frac{1}{\Omega(s)} \; e^{iks(z-z_2)}$$

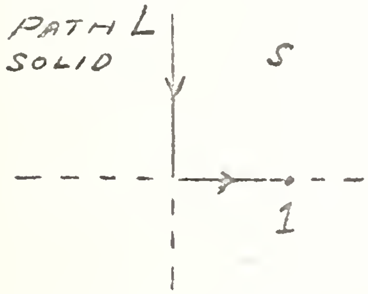
All these functions are one-valued functions of $s = \sqrt{1-\rho^2}$. The only singularities are poles, given by the roots of the transcendental equation

$$\Omega(s) = 0.$$

The function $V(x,y,z)$ itself is given by the integral (7.21). We introduce the variable $s = \sqrt{1-\rho^2}$ instead of ρ and obtain the solution of our problem by the integral

$$V(x,y,z) = \frac{2}{\varepsilon_0}\int_L v(z,s)\; J_0(kr\sqrt{1-s^2})s\; ds$$

where $v(z,s)$, defined in formula (7.23) is given by formula (7.30). The integrand is a one-valued and meromorphic function of s with poles as singularities only. The path of integration in the complex s-plane is shown in the figure. Any deformation of the path must consider the possible poles of the integrand. Of special interest are the values of $V(x,y,z)$ at the boundary surfaces. Here we have to insert in our integral the functions



$$z = 0;\; v(o,s) = \frac{1}{\Omega(s)}\Big\{\cos k\,\gamma_0\ell_0 \cos k\,\gamma_1\ell_1 - \frac{\varepsilon_0}{\varepsilon_1}\frac{\gamma_1}{\gamma_0}\sin k\,\gamma_0\ell_0 \sin k\,\gamma_1\ell_1$$

$$-is\Big(\frac{\varepsilon_1}{\gamma_1}\sin k\,\gamma_1\ell_1 \cos k\,\gamma_0\ell_0 + \frac{\varepsilon_0}{\gamma_0}\cos k\,\gamma_1\ell_1 \sin k\,\gamma_0\ell_0\Big)\Big\}$$

$$z = z_1;\; v(z_1,s) = \frac{1}{\Omega(s)}\Big(\cos k\,\gamma_1\ell_1 - is\,\frac{\varepsilon_1}{\gamma_1}\sin k\,\gamma_1\ell_1\Big)$$

$$z = z_2;\; v(z_2,s) = \frac{1}{\Omega(s)}$$

In particular, we obtain the function $V(x,y,z)$ at the upper boundary by

$$V(x,y,z_2) = \frac{2}{\varepsilon_0}\int_L \frac{1}{\Omega(s)}\; J_0(kr\sqrt{1-s^2})\; s\; d\,s$$

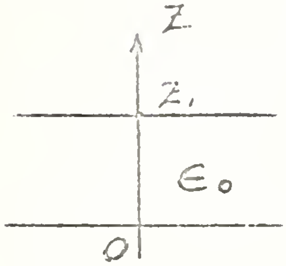
The electromagnetic field, finally, is found by applying the formulas (5.3) with $W = 0$ since our source is a dipole.

$$\varepsilon E_1 = V_{xz} \qquad H_1 = -ikV_y$$

$$\varepsilon E_2 = V_{yz} \qquad H_2 = ikV_x$$

$$\varepsilon E_3 = -(V_{xx} + V_{yy}) \qquad H_3 = 0$$

With our results we may express the solution for the simpler case where the medium consists only of two strata separated by the plane $z = z_1$. We obtain



the solution by assuming $\ell_1 = 0$, i.e., by introducing in the integral

$$V(x,y,z) = \frac{2}{\varepsilon_o}\int_L v(z,s)\, J_o(kr\sqrt{1-s^2}\,)\, s\, ds$$

the functions

$$0 < z < z_1;\ v(z,s) = \frac{1}{\Omega(s)}\left(\cos k\,\gamma_o(z-z_1) + is\,\frac{\varepsilon_o}{\gamma_o}\,\sin k\,\gamma_o(z-z_1)\right)$$

$$z_1 < z;\ v(z,s) = \frac{1}{\Omega(s)}\, e^{ik\,s(z-z_1)} \tag{7.31}$$

where $\Omega(s) = s\cos k\,\gamma_o\,\ell_o - i\,\frac{\gamma_o}{\varepsilon_o}\sin k\,\gamma_o\,\ell_o$

i.e.,

$$V(x,y,z)\begin{cases} = \dfrac{2}{\varepsilon_o}\displaystyle\int_L \dfrac{\cos k\,\gamma_o(z-z_1) + i\,s\,\dfrac{\varepsilon_o}{\gamma_o}\sin k\,\gamma_o(z-z_1)}{s\cos k\,\gamma_o\,\ell_o - i\,\dfrac{\gamma_o}{\varepsilon_o}\sin k\,\gamma_o\,\ell_o}\, J_o(kr\sqrt{1-s^2})\,s\,ds \\[2em] = \dfrac{2}{\varepsilon_o}\displaystyle\int_L \dfrac{J_o(kr\sqrt{1-s^2}\; e^{ik\,s(z-z_1)})}{s\cos k\,\gamma_o\,\ell_o - i\,\dfrac{\gamma_o}{\varepsilon_o}\sin k\,\gamma_o\,\ell_o}\, s\,ds \end{cases} \tag{7.32}$$

where the first expression is valid for $0 < z \leq z_1$, the second for $z > z_1$.

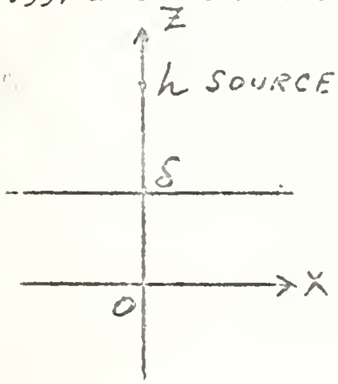
Articles 7.4 assumed that the dipole was on a perfectly conducting ground and therefore utilized equation (7.15). However, in view of the discussion preceding this formula it would be equally possible to consider the case of the dipole above any sort of ground.

7.5 Example of a Stratified Medium with Continuously Varying Index of Refraction.

The preceding article showed how the general theory of an arbitrary source located between two semi-infinite stratified media could be specialized to the case of a dipole located on a perfectly conducting ground with the medium above ground consisting of parallel layers each with constant ε, μ, and σ. The present article will indicate how the general theory can be specialized to the case of a continuously varying dielectric constant. For the purposes of illustration we consider the stratified medium in which $\mu = 1$ and ε varies according to the expression

$$\varepsilon = 1 + \frac{p}{k^2 z^2} \tag{7.33}$$

where p is a positive or negative constant. We assume that our medium is bounded at $z = \delta > 0$ by a homogeneous medium ε_3, μ_3, so that the singularity of formula (7.33) at $z = 0$ is excluded. Since δ and p are arbitrary constants we can approximate any inhomogeneous medium in which ε increases or decreases monotonically by the formula (7.33) with the aid of the two parameters p and δ. For example, if the values ε_1 and ε_2 at the bottom $z = \delta$ and at the height $z = H + \delta$ are known, we can represent the medium by formula (7.33), using the parameters



$$\delta = \frac{H}{\sqrt{\dfrac{\varepsilon_1 - 1}{\varepsilon_2 - 1}} - 1} \tag{7.34}$$

$$p = \frac{(\varepsilon_1 - 1)\, H^2 k^2}{\left(\sqrt{\dfrac{\varepsilon_1 - 1}{\varepsilon_2 - 1}} - 1\right)^2}$$

We assume that a source is located at $z = h > \delta$ whose radiation pattern is known if the surrounding medium is homogeneous. In particular we shall assume that the function V of the two characteristic functions V and W is identically zero, so thatthe electromagnetic field has the special form

$$\begin{aligned} E_1 &= ik\, W_y & H_1 &= W_{xz} \\ E_2 &= -ik\, W_x & H_2 &= W_{yz} \\ E_3 &= 0 & H_3 &= -W_{xx} - W_{yy} \end{aligned} \tag{7.35}$$

It is clear that this is a very special class of transmitters. We choose this class because the differential equation for W admits simple solutions under the assumption (7.33).

According to formula (5.7) we thus consider sources which have, in a homogeneous medium, the radiation pattern $V \equiv 0$ and

$$\text{for } z > h: \quad W_1 = \iint w_1(\alpha, \beta, z)\, e^{ik(\alpha x + \beta y)}\, d\alpha\, d\beta\,. \tag{7.36}$$

$$\text{for } z < h: \quad W_2 = \iint w_2(\alpha, \beta, z)\, e^{ik(\alpha x + \beta y)}\, d\alpha\, d\beta$$

where

$$w_1 = G_1(\alpha, \beta)e^{ik(z-h)\sqrt{n_o^2 - \alpha^2 - \beta^2}}$$

$$w_2 = G_2(\alpha, \beta)e^{-ik(z-h)\sqrt{n_o^2 - \alpha^2 - \beta^2}}$$

where $n_o^2 = \varepsilon_o \mu_o$, ε_o and μ_o being characteristic of the homogeneous medium. G_1 and G_2 being given functions, the modifications of formula (5.7) are required only because the source is at $z = h$. From their very definitions it follows that w_2 and w_2 satisfy at $z = h$ the following characteristic discontinuity conditions;

$$\begin{aligned} w_1(h) - w_2(h) &= G_1 - G_2 \\ w_1'(h) - w_2'(h) &= ik\sqrt{n_o^2 - \rho^2}\,(G_1 + G_2) \end{aligned} \tag{7.37}$$

By these conditions, and by the conditions

$$\frac{1}{\mu_o}\, w_1'(z) - i\,\frac{k}{\mu_o}\sqrt{n_o^2 - \rho^2}\; w_1(z) \to 0 \quad \text{if } z \to \infty \tag{7.38}$$

$$\frac{1}{\mu_o}\, w_2'(z) + i\,\frac{k}{\mu_o}\sqrt{n_o^2 - \rho^2}\; w_2(z) \to 0 \text{ if } z \to -\infty$$

which follow from the results (7.11) where $\theta(z) = \frac{1}{ik}\,\frac{w'(z)}{\mu w(z)}$ (cf. formula (6.4)),and which must hold also when the medium is homogeneous, the solutions $w_1(z)$ and $w_2(z)$ of the ordinary differential equation (See formula (5.5))

$$\mu_o\left(\frac{w'}{\mu_o}\right)' + k^2(n_o^2 - \rho^2)\, w = 0 \tag{7.39}$$

are uniquely determined, if the medium is homogeneous. We thus may characterize the radiation pattern of our source by the discontinuity conditions (7.37) and the boundary conditions (7.38).

Had we considered the class of transmitters for which $W = 0$ and V given by an expression such as (7.36), a class which includes the vertical dipole, we should be led to an equation like (7.39) except that μ would be replaced by ε. Since, however, ε is variable whereas μ is constant, the solution of the analogue to equation (7.39) becomes much more difficult. This case has not been treated as yet.

If, now, the medium is inhomogeneous, as in formula (7.33), we obtain the electromagnetic field by the formulas (7.36) where, however, $w_1(z)$ and $w_2(z)$ are solutions of the differential equation (5.5), namely,

$$w'' + k^2 (\varepsilon - \rho^2) w = 0 \tag{7.40}$$

in which ε is no longer constant but

$$\varepsilon = 1 + \frac{p}{k^2 z^2} \quad ,$$

and wherein we have taken $\mu = 1$.

$w_1(z)$ and $w_2(z)$ must satisfy the conditions

$$w_1'(z) - ik\sqrt{1-\rho^2}\; w_1(z) \to 0 \qquad \text{if } z \to \infty \tag{7.41}$$

$$w_2'(\delta) + ik\mu_3\sqrt{n_3^2 - \rho^2}\; w_2(\delta) = 0 \quad \text{at } z = \delta .$$

where $n_3^2 = \varepsilon_3 \mu_3$ is determined by the material constants ε_3, μ_3 of the boundary medium $z < \delta$. The first condition is merely a restatement of the boundary condition (7.11) in terms of $w_1(z)$ instead of $\Theta_1(z)$. The second condition expresses the fact that the $\Theta(z)$ in the region $\delta < z < h$ must be continuous with the $\Theta(z)$ in the semi-infinite region $z < \delta$. Now, in the latter, $\Theta(z) = \Theta_2(z) = -\frac{\gamma_3}{\mu_3}$, by one of the conditions (7.11). Hence the former $\Theta(z) = \frac{w'(z)}{ik\mu w(z)}$ must equal $-\frac{\gamma_3}{\mu_3}$ at $z = \delta$. In particular $w_2(\delta) = 0$, if the medium $z < \delta$ is a perfect conductor with $\varepsilon_3 = \infty$. (See article 7.2).

The conditions (7.37) remain unchanged in our medium: This expresses the fact that the same source is placed in our inhomogeneous medium.

Our problem is to determine two solutions w_1 and w_2 of equation (7.40), which satisfy the boundary conditions (7.41) and the discontinuity conditions (7.37). For simplicity let us assume that the source is symmetrical, so that c

$G_1 = G_2 = G(\alpha, \beta) = G(\rho)$ where, as usual, $\rho^2 = \alpha^2 + \beta^2$. Let us also assume that the medium $z < \delta$ is a perfect conductor. Then our problem is to find two solutions $w_1(z)$ and $w_2(z)$ of the equation (7.40), namely,

$$w''(z) + k^2 \left(1 + \frac{p}{k^2 z^2} - \rho^2\right) w = 0 \tag{7.42}$$

which satisfy the conditions

$$\begin{aligned} & w_1'(z) - ik\sqrt{1 - \rho^2}\, w_1(z) \to 0, \text{ if } z \to \infty \\ & w_2(\delta) = 0 \end{aligned} \tag{7.43}$$

and the conditions at $z = h$

$$\begin{aligned} & w_1(h) - w_2(h) = 0 \\ & w_1'(h) - w_2'(h) = 2\, ik\sqrt{\varepsilon_0 - \rho^2}\, G(\rho) \end{aligned} \tag{7.44}$$

where

$$\varepsilon_0 = 1 + \frac{p}{k^2 h^2}$$

The general solutions of equation (7.42) are given by the Bessel functions and Hankel functions

$$\sqrt{z}\; J_\nu \left(k\sqrt{1-\rho^2}\; z\right) \text{ and } \sqrt{z}\; H_\nu \left(k\sqrt{1-\rho^2}\; z\right)$$

as one easily verifies. The index, ν, is determined by the relation

$$p = 1/4 - \nu^2$$

i.e.

$$\nu = \pm\sqrt{1/4 - p} \tag{7.45}$$

We satisfy the conditions (7.43) by choosing

$$w_1(z) = C_1 \sqrt{z}\; H_\nu^1 \left(k z \sqrt{1 - \rho^2}\right) \tag{7.46}$$

$$w_2(z) = C_2 \sqrt{z} \left(\frac{H_\nu^2\left(k\sqrt{1-\rho^2}\,\delta\right)}{H_\nu^1(\delta)}\, H_\nu^1\left(k\sqrt{1-\rho^2}\; z\right) - H_\nu^2\left(k\sqrt{1-\rho^2}\; z\right)\right)$$

where C_1 and C_2 are arbitrary constants.

These constants will be determined by the conditions (7.44). We introduce temporarily the notation

$$\Omega_1(z) = H^1_\nu\ (k\,z\sqrt{1-\rho^2}\,)$$

$$\Omega_2(z) = \frac{H^2_\nu\,(k\,\delta\sqrt{1-\rho^2}\,)}{H^1_\nu\,(k\,\delta\sqrt{1-\rho^2}\,)}\ H^1_\nu\,(k\,z\sqrt{1-\rho^2}\,) - H^2_\nu\,(k\,z\sqrt{1-\rho^2}\,)\ .$$

Then to meet the first condition of (7.44) we write $w_1(z)$ and $w_2(z)$ in the form

$$w_1(z) = C\sqrt{zh}\ \Omega_1(z)\Omega_2(h)$$

$$w_2(z) = C\sqrt{zh}\ \Omega_1(h)\,\Omega_2(z) \qquad (7.47)$$

where C is still arbitrary. The second condition of (7.44)

$$w_1'(h) - w_2'(h) = C\ h\Big(\Omega_1'(h)\Omega_2(h) - \Omega_2'(h)\Omega_1(h)\Big)$$

$$= 2\ ik\sqrt{\varepsilon_0-\rho^2}\quad G(\rho) \qquad (7.48)$$

then yields C. With the aid of the identity[20]

$$x\Big(H^1_\nu\,(x)\,\frac{d}{dx}\ H^2_\nu(x) - H^2_\nu(x)\,\frac{d}{dx}\ H^1_\nu(x)\Big) = -\ \frac{4i}{\pi}$$

applied to the first of equations (7.48) we obtain

$$\frac{-4i}{\pi}\,C = 2\ ik\ \sqrt{\varepsilon_0-\rho^2}\ \ G(\rho)$$

i.e.

$$C = \frac{\pi k}{2}\sqrt{\varepsilon_0-\rho^2}\ \ G(\rho)\ ,$$

and hence

$$w_1(z) = -\ \frac{\pi k}{2}\sqrt{\varepsilon_0-\rho^2}\ \ G(\rho)\ \sqrt{zh}\,\Omega_1(z)\,\Omega_2(h)$$

$$w_2(z) = -\ \frac{\pi k}{2}\sqrt{\varepsilon_0-\rho^2}\ \ G(\rho)\sqrt{zh}\,\Omega_1(h)\,\Omega_2(z) \qquad (7.49)$$

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20. See Watson, G. N.: Theory of Bessel Functions, p. 76, formula (4).

The case $G(\rho) = \dfrac{1}{2\pi\sqrt{\varepsilon_0 - \rho^2}}$ is of special interest since, for a homogeneous medium $\varepsilon = \varepsilon_0$ and $\mu_0 = 1$, the equations (7.36) yield the function (see formula 5.9)

$$W = \frac{e^{ik\sqrt{\varepsilon_0}\,r}}{ikr} .$$

Such a source thus gives in our inhomogeneous medium the functions

$$w_1(z) = -\frac{k}{4}\sqrt{zh}\,\Omega_1(z)\Omega_2(h)$$

$$w_2(z) = -\frac{k}{4}\sqrt{zh}\,\Omega_2(z)\Omega_1(h) \qquad (7.50)$$

or, explicitly, in view of the meaning of $\Omega_1(z)$ and $\Omega_2(z)$,

$$w_1(z) = \frac{k}{4}\sqrt{zh}\left(H^2_\nu(kh\sqrt{1-\rho^2}) - \Gamma(\rho)\,H^1_\nu(kh\sqrt{1-\rho^2})\right)H^1_\nu(kz\sqrt{1-\rho^2})$$

$$w_2(z) = \frac{k}{4}\sqrt{zh}\left(H^2_\nu(kz\sqrt{1-\rho^2}) - \Gamma(\rho)\,H^1_\nu(kz\sqrt{1-\rho^2})\right)H^1_\nu(kh\sqrt{1-\rho^2}) \qquad (7.51)$$

where

$$\Gamma(\rho) = \frac{H^2_\nu(k\delta\sqrt{1-\rho^2})}{H^1_\nu(k\delta\sqrt{1-\rho^2})}$$

The function $W(x,y,z)$ itself finally follows from

$$W_1 = \iint w_1(z,\rho)\,e^{ik(\alpha x + \beta y)}\,d\alpha\,d\beta \qquad (7.52)$$

for $z > h$ and from

$$W_2 = \iint w_2(z,\rho)\,e^{ik(\alpha x + \beta y)}\,d\alpha\,d\beta$$

for $z < h$.

The argument for this result is precisely the argument for formula (6.13). However, in this article the theory deals directly with $w(z)$ instead of replacing an expression in w by $\theta(z)$. From formula (7.52) the electromagnetic field can be determined by means of the relations (7.35).

Our result becomes quite simple if $\delta \longrightarrow 0$. This limit will be obtained if we assume that $\nu > 0$, i.e., that $p < 1/4$, for only in case $\nu > 0$ is a definite limit assured for $\Gamma(\rho)$, which alone contains the quantity δ.

In order to find this limit the relations which follow directly from the definition of H^1_ν and H^2_ν, namely,

$$\begin{aligned} 2\, J_{-\nu}(x) &= e^{i\nu\pi} H^1_\nu(x) + e^{-i\nu\pi} H^2_\nu(x) \\ 2\, J_\nu(x) &= H^1_\nu(x) + H^2_\nu(x) \end{aligned} \qquad (7.53)$$

are used. Since J_ν approaches 0 as z^ν it follows that

$$\frac{J_\nu(x)}{J_{-\nu}(x)} = \frac{1 + \Gamma}{e^{i\nu\pi} + e^{-i\nu\pi}\,\Gamma}$$

The limit of the left side as x approaches 0 is zero if $\nu > 0$. Hence $\Gamma \longrightarrow -1$.

Hence as δ approaches 0 the formulas (7.51) thus become, in view of the definition of H^1_ν and H^2_ν ,

$$\begin{aligned} w_1(z) &= \frac{k}{2}\sqrt{zh}\; J_\nu\left(k\,h\sqrt{1-\rho^2}\right) H^1_\nu\left(k\,z\sqrt{1-\rho^2}\right) \\ w_2(z) &= \frac{k}{2}\sqrt{zh}\; J_\nu\left(k\,z\sqrt{1-\rho^2}\right) H^1_\nu\left(k\,h\sqrt{1-\rho^2}\right) \end{aligned} \qquad (7.54)$$

If we substitute these results in formula (7.52) and take advantage of the steps already demonstrated in going from formula (7.20) to formula (7.21) - the quantity $w_1(z)$ here plays the part of $\frac{1}{\Theta_0} e^{ik\int_0^z \varepsilon\,\Theta(z)\,dz}$ there - we obtain

For $z > h$

$$W_1(x,y,z) = k\pi\sqrt{zh}\int_0^\infty J_\nu\left(kh\sqrt{1-\rho^2}\right)H_\nu^1\left(kz\sqrt{1-\rho^2}\right)J_o(kr\rho)\,\rho\,d\rho$$

and for $z < h$ (7.55)

$$W_2(x,y,z) = k\pi\sqrt{zh}\int_0^\infty J_\nu\left(kz\sqrt{1-\rho^2}\right)H_\nu^1\left(kh\sqrt{1-\rho^2}\right)J_o(kr\rho)\,\rho\,d\rho$$

where $r = \sqrt{x^2+y^2}$.

This result can perhaps be further simplified by the use of properties of Bessel functions. However, this will not be done here, for the primary purpose of this article is to indicate how the general theory specializes for the case of a medium with a continuously varying index of refraction (or dielectric constant). We are not at the moment interested so much in the particular law of variation nor of the particular class of transmitters involved as in exploring the type of integrals to which the general theory leads in particular cases. Application of the above theory to investigate laws of variation of the ε or of the index of refraction which are actually found in the atmosphere hinges upon the direction of future research as discussed in the introduction to this paper.